

FRÖLICHER-NIJENHUIS COHOMOLOGY ON G_2 - AND $\text{Spin}(7)$ -MANIFOLDS

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ABSTRACT. In this paper we show that a parallel differential form Ψ of even degree on a Riemannian manifold allows to define a natural differential both on $\Omega^*(M)$ and $\Omega^*(M, TM)$, defined via the Frölicher-Nijenhuis bracket. For instance, on a Kähler manifold, these operators are the complex differential and the Dolbeault differential, respectively. We investigate this construction when taking the differential w.r.t. the canonical parallel 4-form on a G_2 - and $\text{Spin}(7)$ -manifold, respectively. We calculate the cohomology groups of $\Omega^*(M)$ and give a partial description of the cohomology of $\Omega^*(M, TM)$.

CONTENTS

1. Introduction	2
2. Preliminaries	4
2.1. The Hodge-* operator	4
2.2. Graded Lie algebras and differentials	5
2.3. The Frölicher-Nijenhuis bracket	6
2.4. Riemannian manifolds with parallel forms	7
3. The Frölicher-Nijenhuis cohomology of G_2 -manifolds	14
4. The Frölicher-Nijenhuis cohomology of $\text{Spin}(7)$ -manifolds	24
5. Deformations of G_2 - and $\text{Spin}(7)$ -structures	27
6. Functorial properties of Frölicher-Nijenhuis-cohomology	31
7. Appendix	32
Acknowledgement	34
References	34

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1. INTRODUCTION

In Riemannian geometry, the investigation of manifolds with special holonomy is a central field of research. Assuming a Riemannian manifold is irreducible and not locally symmetric, there are only few possible subgroups of the orthogonal group which can occur as holonomies, classified by Berger [Berger1955]. Most of the possible holonomy groups may be characterized as the stabilizer of (one or two) differential forms in the orthogonal group, leading to the existence of parallel forms on such manifolds. For instance, Kähler manifolds admit the parallel Kähler-2-form, Calabi-Yau manifolds admit in addition a complex volume form [Yau1978]; hyper-Kähler manifolds admit three linearly independent parallel 2-forms [Calabi1979], and similarly, manifolds with exceptional holonomy G_2 in dimension 7 and $\text{Spin}(7)$ in dimension 8 may be characterized by a parallel 4-form [Bonan1966], [Gray1971]. For the extremely rich structure of manifolds with these holonomy groups, we refer to [HL1982], [Bryant1987], [Bryant2005], [BS1989], and for the discussion of closed manifolds with these holonomies we refer to [Joyce1996a], [Joyce1996b], [Joyce2000] and [Joyce2007]. We also refer to [Le2013] for description of Riemannian manifolds admitting a parallel 3-form.

It is the aim of the present paper to construct cohomology invariants on an oriented Riemannian manifold (M, g) with a parallel form of even degree. Namely, to such a form, say Ψ^{2k} , we associate two differentials

$$\mathcal{L}_{\Psi^{2k}} : \Omega^*(M) \rightarrow \Omega^{*+2k-1}(M), \quad \mathcal{L}_{\Psi^{2k}} : \Omega^*(M, TM) \rightarrow \Omega^{*+2k-1}(M, TM),$$

i.e., $\mathcal{L}_{\Psi^{2k}}$ is a derivation and $\mathcal{L}_{\Psi^{2k}} \circ \mathcal{L}_{\Psi^{2k}} = 0$ in each case. Their cohomologies, denoted as $H_{\Psi^{2k}}^*(M)$ and $H_{\Psi^{2k}}^*(M, TM)$, respectively, form a graded algebra and a graded Lie algebra, respectively, where the i -th cohomologies are defined as

$$(1.1) \quad \begin{aligned} H_{\Psi^{2k}}^i(M) &:= \frac{\ker \mathcal{L}_{\Psi^{2k}} : \Omega^i(M) \rightarrow \Omega^{i+2k-1}(M)}{\Im \mathcal{L}_{\Psi^{2k}} : \Omega^{i-2k+1}(M) \rightarrow \Omega^i(M)}, \\ H_{\Psi^{2k}}^i(M, TM) &:= \frac{\ker \mathcal{L}_{\Psi^{2k}} : \Omega^i(M, TM) \rightarrow \Omega^{i+2k-1}(M, TM)}{\Im \mathcal{L}_{\Psi^{2k}} : \Omega^{i-2k+1}(M, TM) \rightarrow \Omega^i(M, TM)}. \end{aligned}$$

The action of $\mathcal{L}_{\Psi^{2k}}$ on differential forms is given by the formula

$$\mathcal{L}_{\Psi^{2k}}(\alpha) = (d^* \alpha) \wedge \Psi^{2k} - d^*(\alpha \wedge \Psi^{2k}).$$

For instance, in the case of a Kähler manifold, using the Kähler form $\Psi = \omega$, the differential \mathcal{L}_ω on $\Omega^*(M)$ is the complex differential $d^c := i(\partial - \bar{\partial})$, whereas on $\Omega^*(M, TM)$ it coincides with the Dolbeault differential $\bar{\partial} : \Omega^{p,q}(M, TM) \rightarrow \Omega^{p,q+1}(M, TM)$ [FN1956b], cf. Example 2.4. Thus, these differentials recover well known and natural cohomology theories.

On the other hand, for G_2 - and $\text{Spin}(7)$ -manifolds, there are canonical parallel 4-forms, denoted by $*\varphi$ and Φ , respectively, and we may consider the respective differentials $\mathcal{L}_{*\varphi}$ and \mathcal{L}_Φ . On closed manifolds, we obtain the following results on their cohomology groups.

Theorem 1.1. *Let (M^7, φ) be a closed G_2 -manifold. Then for the cohomologies $H_{*\varphi}^i(M^7)$ and $H_{*\varphi}^i(M^7, TM^7)$ defined above, the following hold.*

(1) *There is a Hodge decomposition*

$$\begin{aligned} H_{*\varphi}^i(M^7) &= \mathcal{H}^i(M^7) \oplus (H_{*\varphi}^i(M^7) \cap d\Omega^{i-1}(M^7)) \\ &\quad \oplus (H_{*\varphi}^i(M^7) \cap d^*\Omega^{i+1}(M^7)), \end{aligned}$$

where $\mathcal{H}^i(M^7)$ denotes the spaces of harmonic forms.

- (2) *The Hodge-* induces an isomorphism $*$: $H_{*\varphi}^i(M^7) \rightarrow H_{*\varphi}^{7-i}(M^7)$.*
- (3) *$H_{*\varphi}^i(M^7) = \mathcal{H}^i(M^7)$ for $i = 0, 1, 6, 7$. For $i = 2, 3, 4, 5$, $H_{*\varphi}^i(M^7)$ is infinite dimensional.*
- (4) *$\dim H_{*\varphi}^0(M^7, TM^7) = b^1(M^7)$; in particular, $H_{*\varphi}^0(M^7) = 0$ if M^7 has full holonomy G_2 .*
- (5) *$\dim H_{*\varphi}^3(M^7, TM^7) \geq b^3(M^7) > 0$, as it contains all torsion free deformations of the G_2 -structure modulo deformations by diffeomorphisms.*

In fact, in Theorem 3.5 we shall give a precise description of the cohomology ring $H_{*\varphi}^*(M^7)$. Analogously, for $\text{Spin}(7)$ -manifolds we have

Theorem 1.2. *Let (M^8, Φ) be a closed $\text{Spin}(7)$ -manifold. Then for the cohomologies $H_\Phi^i(M^8)$ and $H_\Phi^i(M^8, TM^8)$ defined above, the following hold.*

(1) *There is a Hodge decomposition*

$$\begin{aligned} H_\Phi^i(M^8) &= \mathcal{H}^i(M^8) \oplus (H_\Phi^i(M^8) \cap d\Omega^{i-1}(M^8)) \\ &\quad \oplus (H_\Phi^i(M^8) \cap d^*\Omega^{i+1}(M^8)), \end{aligned}$$

where $\mathcal{H}^i(M^8)$ denotes the spaces of harmonic forms.

- (2) *The Hodge-* induces an isomorphism $*$: $H_\Phi^i(M^8) \rightarrow H_\Phi^{8-i}(M^8)$.*
- (3) *$H_\Phi^i(M^8) = \mathcal{H}^i(M^8)$ for $i = 0, 1, 2, 6, 7, 8$. For $i = 3, 4, 5$, $H_\Phi^i(M^8)$ is infinite dimensional.*
- (4) *$\dim H_\Phi^0(M^8, TM^8) = b^1(M^8)$; in particular, $H_\Phi^0(M^8) = 0$ if M^8 has full holonomy $\text{Spin}(7)$.*
- (5) *$\dim H_\Phi^3(M^8, TM^8) \geq b_1^4(M^8) + b_7^4(M^8) + b_{35}^4(M^8) > 0$, as it contains all torsion free deformations of the $\text{Spin}(7)$ -structure modulo deformations by diffeomorphisms.*

In fact, in Theorem 4.2 we shall give a precise description of the cohomology ring $H_\Phi^*(M^8)$.

In order to define these cohomologies, we utilize the *Frölicher-Nijenhuis bracket* $[\cdot, \cdot]^{FN}$ which provides the graded space $\Omega^*(M, TM)$ with the structure of a graded Lie algebra, acting on the graded algebra $\Omega^*(M)$ by graded derivations, say, $\mathcal{L}_K : \Omega^*(M) \rightarrow \Omega^*(M)$ for $K \in \Omega^*(M, TM)$, and on $\Omega^*(M, TM)$ by the adjoint ad_K [FN1956]. If Ψ^{2k} is a parallel form and $\hat{\Psi} := \partial_g \Psi^{2k} \in \Omega^{2k-1}(M, TM)$ is the contraction with the metric g , then $[\hat{\Psi}^{2k}, \hat{\Psi}^{2k}]^{FN} = 0$, which implies that $\mathcal{L}_{\hat{\Psi}^{2k}}$ (which is also denoted by $\mathcal{L}_{\Psi^{2k}}$) is a differential, so its cohomology can be defined as above.

If the underlying manifold is closed, then this cohomology has a Hodge decomposition and contains all harmonic forms, as will be shown in Theorems 1.1 and 1.2. If the differential \mathcal{L}_Ψ satisfies some regularity condition, then we show that the Hodge-duality of the cohomology holds. As it turns out, this regularity condition is satisfied for the parallel 4-forms on G_2 - and $\text{Spin}(7)$ -manifolds, respectively, allowing us to give the explicit descriptions of the cohomology in Theorems 1.1 and 1.2.

Our paper is organized as follows. In Section 2 we provide the basic definitions of graded Lie algebras and the Frölicher-Nijenhuis bracket, and show that a parallel form of even degree on a Riemannian manifold yields a differential on both $\Omega^*(M)$ and $\Omega^*(M, TM)$ which has the properties asserted above for closed M . In Section 3 we apply this to G_2 -manifolds, giving explicit descriptions of the cohomology $H_{*\varphi}^*(\Omega^*(M))$ and repeat this in Section 4 for $\text{Spin}(7)$ -manifolds. In Section 5 we discuss the relation of the cohomology groups $H^*(M, TM)$ and deformations, and show the results on the cohomologies $H_{*\varphi}^*(M^7, TM^7)$ and $H_{\Phi}^*(M^8, TM^8)$, respectively. In Section 6 we collect some functorial properties of homology defined by Frölicher-Nijenhuis bracket. Finally, we outsourced some technical results into the appendix.

2. PRELIMINARIES

2.1. The Hodge-* operator. Let (V, g) be an n -dimensional oriented real vector space with a scalar product g and volume form vol . Then the *musical operators* allow to identify V and V^* by

$$\flat : V \longrightarrow V^*, \quad v \longmapsto v^\flat := g(v, \cdot)$$

and

$$\sharp := \flat^{-1} : V^* \longrightarrow V, \quad \alpha \longmapsto \alpha^\sharp.$$

g also induces an inner product on $\Lambda^k V^*$ in a canonical way, and the *Hodge-* operator* $* : \Lambda^k V^* \rightarrow \Lambda^{n-k} V^*$ is the unique linear map satisfying the identity

$$\alpha^k \wedge \beta^{n-k} = g(*\alpha^k, \beta^{n-k}) \text{vol} = g(*\alpha^k, \beta^{n-k}) * 1$$

for all $\alpha^k \in \Lambda^k V^*$ and $\beta^{n-k} \in \Lambda^{n-k} V^*$. In particular,

$$(2.1) \quad *^2|_{\Lambda^k V^*} = (-1)^{k(n-k)} \text{Id}_{\Lambda^k V^*} \quad \text{and} \quad g(*\alpha^k, *\beta^k) = g(\alpha^k, \beta^k).$$

Moreover, for $\alpha^k \in \Lambda^k V^*$ and $v \in V$, we have the relation

$$(2.2) \quad \iota_v(*\alpha^k) = (-1)^k * (v^\flat \wedge \alpha^k) \quad \text{and} \quad *(\iota_v \alpha^k) = (-1)^{k+1} v^\flat \wedge *\alpha^k,$$

where ι_v denotes the contraction of the form α^k with the vector v .

Let (M, g) be an n -dimensional oriented Riemannian manifold and let ∇ denotes its Levi-Civita connection. Then ∇ is compatible with $*$, i.e.,

$$*(\nabla_v \alpha^k) = \nabla_v (*\alpha^k).$$

A (local) oriented orthonormal frame (e_i) with dual (local) coframe $(e^i) = (e_i^\flat)$ is called *normal in* $p \in M$ if $\nabla e_i|_p = 0$ or, equivalently, $\nabla e^i|_p = 0$ for all

i. The existence of such a (co-)frame can be shown e.g. by choosing normal coordinates (x^i) around $p = 0$ and letting $e_i := \sqrt{g}^{ij} \partial_j$. If (e_i) is normal in p then the exterior derivative d and its formal dual d^* at p are given as

$$(2.3) \quad d\alpha^k|_p = e^i \wedge \nabla_{e_i} \alpha^k, \quad d^* \alpha^k|_p = (-1)^{n(n-k)+1} * d * \alpha^k = -\iota_{e_i} \nabla_{e_i} \alpha^k.$$

Finally, for later reference, we also recall the map defined by contraction of a form with the metric g , i.e.

$$(2.4) \quad \partial = \partial_g : \Lambda^k V^* \longrightarrow \Lambda^{k-1} V^* \otimes V, \quad \partial_g(\alpha^k) := (\iota_{e_i} \alpha^k) \otimes (e^i)^\#,$$

where the sum is taken over some basis (e_i) of V with dual basis (e^i) .

2.2. Graded Lie algebras and differentials. In this section, we briefly recall some basic notions and properties of graded algebras and graded Lie algebras. Let $V := (\bigoplus_{k \in \mathbb{Z}} V_k, \cdot)$ be a graded real vector space with a graded bilinear map $\cdot : V \times V \rightarrow V$, called a *product on V* . A *graded derivation of (V, \cdot) of degree l* is a linear map $D^l : V \rightarrow V$ of degree l (i.e., $D^l(V_k) \subset V_{k+l}$) such that

$$(2.5) \quad D^l(x \cdot y) = (D^l x) \cdot y + (-1)^{|x|} x \cdot (D^l y),$$

where $|x|$ denotes the degree of an element, i.e. $|x| = k$ for $x \in V_k$. If we denote by $\mathcal{D}^l(V, \cdot)$ the graded derivations of (V, \cdot) of degree l , then $\mathcal{D}(V) := \bigoplus_{l \in \mathbb{Z}} \mathcal{D}^l(V)$ is a graded Lie algebra with the Lie bracket

$$(2.6) \quad [D_1, D_2] := D_1 D_2 - (-1)^{|D_1||D_2|} D_2 D_1,$$

i.e., the Lie bracket is graded anti-symmetric and satisfies the graded Jacobi identity,

$$(2.7) \quad [x, y] = -(-1)^{|x||y|} [y, x]$$

$$(2.8) \quad (-1)^{|x||z|} [x, [y, z]] + (-1)^{|y||x|} [y, [z, x]] + (-1)^{|z||y|} [z, [x, y]] = 0.$$

In general, if $L = (\bigoplus_{k \in \mathbb{Z}} L_k, [\cdot, \cdot])$ is a graded Lie algebra, then an *action of L on V* is a Lie algebra homomorphism $\pi : L \rightarrow \mathcal{D}(V)$, which yields a graded bilinear map $L \times V \rightarrow V$, $(x, v) \mapsto \pi(x)(v)$ such that the map $\pi(x) : V \rightarrow V$ is a graded derivation of degree $|x|$ and such that

$$[\pi(x), \pi(y)] = \pi[x, y].$$

For instance, a graded Lie algebra acts on itself via the adjoint representation $ad : L \rightarrow \mathcal{D}(L)$, where $ad_x(y) := [x, y]$.

For a graded Lie algebra L we define the set of *Maurer-Cartan elements of L of degree $2k+1$* as

$$\mathcal{MC}^{2k+1}(L) := \{\xi \in L_{2k+1} \mid [\xi, \xi] = 0\}.$$

If $\pi : L \rightarrow \mathcal{D}(V)$ is an action of L on (V, \cdot) , then for $\xi \in \mathcal{MC}^{2k+1}(L)$ we have $0 = [\pi(\xi), \pi(\xi)] = 2\pi(\xi)^2$, so that $\pi(\xi) : V \rightarrow V$ is a differential on V . We define the *cohomology of (V, \cdot) w.r.t. ξ* as

$$(2.9) \quad H_\xi^i(V) := \frac{\ker(\pi(\xi) : V_i \rightarrow V_{i+2k+1})}{\Im(\pi(\xi) : V_{i-(2k+1)} \rightarrow V_i)} \quad \text{for } \xi \in \mathcal{MC}^{2k+1}(L).$$

Since $\pi(\xi)$ is a derivation, it follows that $\ker \pi(\xi) \cdot \ker \pi(\xi) \subset \ker \pi(\xi)$, whence there is an induced product on $H_\xi^*(V) := \bigoplus_{i \in \mathbb{Z}} H_\xi^i(V)$.

If $L = \bigoplus_{k \in \mathbb{Z}} L_k$ is a graded Lie algebra, then for $v \in L_0$ and $t \in \mathbb{R}$, we define the formal power series

$$(2.10) \quad \exp(tv) : L \longrightarrow L[[t]], \quad \exp(tv)(x) := \sum_{k=0}^{\infty} \frac{t^k}{k!} \text{ad}_v^k(x).$$

Observe that $\text{ad}_\xi(v) = 0$ for some $v \in L_0$ iff $\text{ad}_v(\xi) = 0$ iff $\exp(tv)(\xi) = \xi$ for all $t \in \mathbb{R}$. In this case, we call v an *infinitesimal stabilizer* of ξ .

For $\xi \in \mathcal{MC}^{2k+1}(L)$, we say that $x \in L_{2k+1}$ is an *infinitesimal deformation* of ξ within $\mathcal{MC}^{2k+1}(L)$ if $[\xi + tx, \xi + tx] = 0 \pmod{t^2}$. Evidently, this is equivalent to $[\xi, x] = 0$ or $x \in \ker \text{ad}_\xi$. Such an infinitesimal deformation is called *trivial* if $x = [\xi, v]$ for some $v \in L_0$, since in this case, $\xi + tx = \exp(-tv)(\xi) \pmod{t^2}$, whence up to second order, it coincides with elements in the orbit of ξ under the (formal) action of $\exp(tv)$. Thus, we have the following interpretation of some cohomology groups.

Proposition 2.1. *Let $(L = \bigoplus_{i \in \mathbb{Z}} L_i, [\cdot, \cdot])$ be a real graded Lie algebra, acting on itself by the adjoint representation, and let $\xi \in \mathcal{MC}^{2k+1}(L)$. Then the following holds.*

- (1) *If $L_{-(2k+1)} = 0$, then $H_\xi^0(L)$ is the Lie algebra of infinitesimal stabilizers of ξ .*
- (2) *$H_\xi^{2k+1}(L)$ is the space of infinitesimal deformations of ξ within $\mathcal{MC}^{2k+1}(L)$ modulo trivial deformations.*

2.3. The Frölicher-Nijenhuis bracket. We shall apply our discussion from the preceding section to the following example. Let M be a manifold and $(\Omega^*(M), \wedge) = (\bigoplus_{k \geq 0} \Omega^k(M), \wedge)$ be the graded algebra of differential forms. Evidently, the exterior derivative $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ is a derivation of $\Omega^*(M)$ of degree 1, whereas contraction $\iota_X : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ with a vector field $X \in \mathfrak{X}(M)$ is a derivation of degree -1 .

More generally, for $K \in \Omega^k(M, TM)$ we define $\iota_K \alpha^l$ as the *contraction* of K with $\alpha^l \in \Omega^l(M)$ pointwise by

$$\iota_{\kappa^k \otimes X} \alpha^l := \kappa^k \wedge (\iota_X \alpha^l) \in \Omega^{k+l-1}(M),$$

where $\kappa^k \in \Omega^k(M)$ and $X \in \mathfrak{X}(M)$, and this is a derivation of $\Omega^*(M)$ of degree $k-1$. Thus, the *Nijenhuis-Lie derivative* along $K \in \Omega^k(M, TM)$ defined as

$$(2.11) \quad \mathcal{L}_K(\alpha^l) := [\iota_K, d](\alpha^l) = \iota_K(d\alpha^l) + (-1)^k d(\iota_K \alpha^l) \in \Omega^{k+l}(M)$$

is a derivation of $\Omega^*(M)$ of degree k , and evidently,

$$(2.12) \quad \mathcal{L}_K(d\alpha) = (-1)^k d\mathcal{L}_K(\alpha).$$

Moreover, if $K = \kappa_i^k \otimes e_i$ w.r.t. some local frame (e_i) , then

$$(2.13) \quad \mathcal{L}_K(\alpha^l) := \kappa_i^k \wedge \mathcal{L}_{e_i} \alpha^l + (-1)^k d\kappa_i^k \wedge \iota_{e_i} \alpha^l.$$

Observe that for $k = 0$ in which case $K \in \Omega^0(M, TM)$ is a vector field, both ι_K and \mathcal{L}_K coincide with the standard notion of contraction with and Lie derivative along a vector field.

In [FN1956] [FN1956b], it was shown that $\Omega^*(M, TM)$ carries a unique structure of a graded Lie algebra, defined by the so-called *Frölicher-Nijenhuis bracket*,

$$[\cdot, \cdot]^{FN} : \Omega^k(M, TM) \times \Omega^l(M, TM) \rightarrow \Omega^{k+l}(M, TM)$$

such that \mathcal{L} defines an action of $\Omega^*(M, TM)$ on $\Omega^*(M)$, that is,

$$(2.14) \quad \mathcal{L}_{[K_1, K_2]^{FN}} = [\mathcal{L}_{K_1}, \mathcal{L}_{K_2}] = \mathcal{L}_{K_1} \circ \mathcal{L}_{K_2} - (-1)^{|K_1||K_2|} \mathcal{L}_{K_2} \circ \mathcal{L}_{K_1}.$$

It is given by the following formula for $\alpha^k \in \Omega^k(M)$, $\beta^l \in \Omega^l(M)$, $X_1, X_2 \in \mathfrak{X}(M)$ [KMS1993, Theorem 8.7 (6), p. 70]:

$$(2.15) \quad \begin{aligned} [\alpha^k \otimes X_1, \beta^l \otimes X_2]^{FN} &= \alpha^k \wedge \beta^l \otimes [X_1, X_2] \\ &+ \alpha^k \wedge \mathcal{L}_{X_1} \beta^l \otimes X_2 - \mathcal{L}_{X_2} \alpha^k \wedge \beta^l \otimes X_1 \\ &+ (-1)^k \left(d\alpha^k \wedge (\iota_{X_1} \beta^l) \otimes X_2 + (\iota_{X_2} \alpha^k) \wedge d\beta^l \otimes X_1 \right). \end{aligned}$$

In particular, for a vector field $X \in \mathfrak{X}(M)$ and $K \in \Omega^*(M, TM)$ we have [KMS1993, Theorem 8.16 (5), p. 75]

$$\mathcal{L}_X(K) = [X, K]^{FN},$$

that is, the Frölicher-Nijenhuis bracket with a vector field coincides with the Lie derivative of the tensor field $K \in \Omega^*(M, TM)$. This means that $\exp(tX) : \Omega^*(M, TM) \rightarrow \Omega^*(M, TM)[[t]]$ is the action induced by (local) diffeomorphisms of M . Thus, Proposition 2.1 now immediately implies the following result.

Theorem 2.2. *Let M be a manifold and $K \in \Omega^{2k+1}(M, TM)$ be such that $[K, K]^{FN} = 0$, and define the differential $d_K(K') := [K, K']^{FN}$. Then*

- (1) $H_K^0(\Omega^*(M, TM))$ is the Lie algebra of vector fields stabilizing K .
- (2) $H_K^{2k+1}(\Omega^*(M, TM))$ is the space of infinitesimal deformations of K within the differentials of $\Omega^*(M, TM)$ of the form $\text{ad}_{\xi^{2k+1}}$, modulo (local) diffeomorphisms.

2.4. Riemannian manifolds with parallel forms. Suppose that (M, g) is an n -dimensional Riemannian manifold with Levi-Civita connection ∇ , and suppose that Ψ is a parallel form of even degree. We now make the following simple but crucial observation.

Proposition 2.3. *Let (M, g) be a Riemannian manifold and $\Psi \in \Omega^{2k}(M)$ be a parallel form of even degree, and let $\hat{\Psi} := \partial_g \Psi \in \Omega^{2k-1}(M, TM)$ with the contraction map ∂_g from (2.4).*

Then $\hat{\Psi}$ is a Maurer-Cartan element, i.e., $[\hat{\Psi}, \hat{\Psi}]^{FN} = 0$.

Proof. Choose geodesic normal coordinates (x^i) around $p \in M$ in such a way that $(\partial_i)_p := (\partial/\partial x^i)_p$ is an orthonormal basis of $T_p M$. The dual basis of ∂_i is dx^i , whence $(dx^i)^\# = g^{ij}\partial_j$. Thus,

$$\hat{\Psi} = (\iota_{\partial_i} \Psi) \otimes (dx^i)^\# = g^{ij}(\iota_{\partial_i} \Psi) \otimes \partial_j.$$

Thus, by (2.15)

$$\begin{aligned} [\hat{\Psi}, \hat{\Psi}]^{FN} &= [g^{ij}(\iota_{\partial_i} \Psi) \otimes \partial_j, g^{rs}(\iota_{\partial_r} \Psi) \otimes \partial_s]^{FN} \\ &= (g^{ij}(\iota_{\partial_i} \Psi) \wedge \mathcal{L}_{\partial_j}(g^{rs}(\iota_{\partial_r} \Psi)) \otimes \partial_s \\ &\quad - \mathcal{L}_{\partial_s}(g^{ij}(\iota_{\partial_i} \Psi)) \wedge g^{rs}(\iota_{\partial_r} \Psi) \otimes \partial_j \\ &\quad - d(g^{ij}(\iota_{\partial_i} \Psi)) \wedge \iota_{\partial_j}(g^{rs}(\iota_{\partial_r} \Psi)) \otimes \partial_s \\ &\quad - (\iota_{\partial_s}(g^{ij}(\iota_{\partial_i} \Psi)) \wedge d(g^{rs}(\iota_{\partial_r} \Psi)) \otimes \partial_j). \end{aligned}$$

Since at p , $g_{ij} = g^{ij} = \delta_{ij}$, $\partial_r g_{ij} = 0$, $\mathcal{L}_{\partial_j} \Psi = \nabla_{e_j} \Psi = 0$, $\nabla_{\partial_i} \partial_j = 0$, and $\partial_j = (e^j)^\#$, it follows that $[\hat{\Psi}, \hat{\Psi}]_p^{FN} = 0$, and $p \in M$ was arbitrary. \square

Thus, by the discussion in Section 2.2, the Lie derivative $\mathcal{L}_{\hat{\Psi}}$ and the adjoint map $ad_{\hat{\Psi}}$ are differentials on $\Omega^*(M)$ and $\Omega^*(M, TM)$, respectively, and for simplicity, we shall denote these by

$$\mathcal{L}_\Psi : \Omega^*(M) \longrightarrow \Omega^*(M) \quad \text{and} \quad ad_\Psi : \Omega^*(M, TM) \longrightarrow \Omega^*(M, TM),$$

or, if we wish to specify the degree,

$$(2.16) \quad \begin{aligned} \mathcal{L}_{\Psi;l} : \Omega^{l-2k+1}(M) &\longrightarrow \Omega^l(M) & \text{and} \\ ad_{\Psi;l} : \Omega^{l-2k+1}(M, TM) &\longrightarrow \Omega^l(M, TM). \end{aligned}$$

The cohomology algebras we denote by $H_\Psi^*(M)$ and $H_\Psi^*(TM)$ instead of $H_\Psi^*(\Omega^*(M))$ and $H_\Psi^*(\Omega^*(M, TM))$, respectively. That is,

$$(2.17) \quad H_\Psi^l(M) = \frac{\ker(\mathcal{L}_{\Psi;l+2k-1})}{\Im(\mathcal{L}_{\Psi;l})} \quad \text{and} \quad H_\Psi^l(TM) = \frac{\ker(ad_{\Psi;l+2k-1})}{\Im(ad_{\Psi;l})}.$$

Observe that a priori we do not have any well behaved topology on $H_\Psi^l(M)$ and $H_\Psi^l(TM)$, since it is not clear if the denominators in (2.17) are closed subspaces of the respective numerator in the Frechét topology.

We assert that

$$(2.18) \quad \mathcal{L}_\Psi \alpha^l = (d^* \alpha^l) \wedge \Psi - d^*(\alpha^l \wedge \Psi).$$

In order to see this, observe that just as in the proof of Proposition 2.3 for an orthonormal frame (e_i) which is normal at $p \in M$ (2.13) implies that

$$(2.19) \quad \mathcal{L}_\Psi \alpha^l|_p = (\iota_{e_i} \Psi) \wedge \nabla_{e_i} \alpha^l|_p,$$

and since $\nabla \Psi = 0$ we have

$$(\iota_{e_i} \Psi) \wedge \nabla_{e_i} \alpha^l|_p = \iota_{e_i} \nabla_{e_i} (\Psi \wedge \alpha^l)|_p - \Psi \wedge (\iota_{e_i} \nabla_{e_i} \alpha^l)|_p,$$

from which (2.18) follows at $p \in M$ by (2.3). In particular, (2.18) together with (2.12) implies

$$(2.20) \quad \mathcal{L}_\Psi d\alpha^l = -d\mathcal{L}_\Psi \alpha^l \quad \text{and} \quad \mathcal{L}_\Psi d^* \alpha^l = -d^* \mathcal{L}_\Psi \alpha^l$$

and therefore,

$$(2.21) \quad \mathcal{L}_\Psi \Delta \alpha^l = \Delta \mathcal{L}_\Psi \alpha^l.$$

By the Weitzenböck formula (see e.g. [Besse1987, (1.154)]), it follows that exterior multiplication with a parallel form commutes with the Laplacian. That is, we have for all $\alpha \in \Omega^*(M)$

$$(2.22) \quad \Delta(\alpha \wedge \Psi) = (\Delta \alpha) \wedge \Psi \quad \text{and} \quad \Delta(\alpha \wedge * \Psi) = (\Delta \alpha) \wedge * \Psi.$$

Example 2.4. Let (M, J, g) be a Kähler manifold with Kähler form ω . Then ω is parallel, and the map $L : \Omega^*(M) \rightarrow \Omega^{*+2}(M)$, $\alpha \mapsto \alpha \wedge \omega$ is called the *Lefschetz map*. In this case, $\mathcal{L}_\omega = [L, d^*] = d^c = i(\bar{\partial} - \partial)$ is the *complex differential*, where $d = \partial + \bar{\partial}$ is the decomposition of the exterior differential into its holomorphic and anti-holomorphic part [FN1956b, (4.1), (4.2)]. Likewise, a straightforward calculation using (2.15) shows that the adjoint map $ad_\omega : \Omega^*(M, TM) \rightarrow \Omega^*(M, TM)$ coincides with the *Dolbeault differential* $\bar{\partial} : \Omega^{p,q}(M, TM) \rightarrow \Omega^{p,q+1}(M, TM)$ [FN1956b, Lemma 4]. In particular, the cohomology algebras $H_\omega^*(M)$ and $H_\omega^*(M, TM)$ are finite dimensional if M is closed.

Recall that for a closed oriented Riemannian manifold (M, g) there is the *Hodge decomposition* of differential forms

$$(2.23) \quad \Omega^l(M) = \mathcal{H}^l(M) \oplus d\Omega^{l-1}(M) \oplus d^* \Omega^{l+1}(M),$$

where $\mathcal{H}^l(M) \subset \Omega^l(M)$ denotes the space of harmonic forms. Moreover $\dim \mathcal{H}^l(M) = \dim H_{dR}^l(M) = b^l(M)$ is a topological invariant.

Let us now compute the formal adjoint of \mathcal{L}_Ψ .

Proposition 2.5. *Let (M, g) , $\Psi \in \Omega^{2k}(M)$ and $\hat{\Psi} := \partial_g \Psi \in \Omega^{2k-1}(M, TM)$ be as before. Then the formal adjoint $\mathcal{L}_{\Psi;l}^* : \Omega^l(M) \rightarrow \Omega^{l-2k+1}(M)$ of $\mathcal{L}_{\Psi;l} : \Omega^{l-2k+1}(M) \rightarrow \Omega^l(M)$ is given by*

$$(2.24) \quad \mathcal{L}_{\Psi;l}^* \alpha^l = (-1)^{n(n-l)+1} * \mathcal{L}_\Psi * \alpha^l$$

Observe the analogy of the formula for $\mathcal{L}_{\Psi;l}^*$ in (2.24) and that for d^* in (2.3).

Proof. For $\alpha^l \in \Omega^l(M)$ and $\beta^{l-2k+1} \in \Omega^{l-2k+1}(M)$, we have

$$(2.25) \quad \begin{aligned} \langle \mathcal{L}_\Psi \beta^{l-2k+1}, \alpha^l \rangle_{L^2} &\stackrel{(2.18)}{=} \langle (d^* \beta^{l-2k+1}) \wedge \Psi - d^*(\beta^{l-2k+1} \wedge \Psi), \alpha^l \rangle_{L^2} \\ &= \int_M d^* \beta^{l-2k+1} \wedge \Psi \wedge * \alpha^l - \int_M \beta^{l-2k+1} \wedge \Psi \wedge * d \alpha^l. \end{aligned}$$

Evaluating the two summands, we get by (2.1) and (2.3)

$$\begin{aligned}
& \int_M d^* \beta^{l-2k+1} \wedge \Psi \wedge * \alpha^l \\
& \stackrel{(2.1)}{=} (-1)^{(n-l)l} \langle d^* \beta^{l-2k+1}, *(\Psi \wedge * \alpha^l) \rangle \\
& = (-1)^{(n-l)l} \langle \beta^{l-2k+1}, d * (\Psi \wedge * \alpha^l) \rangle_{L^2} \\
& \stackrel{(2.1),(2.3)}{=} (-1)^{n(n-l)} \langle \beta^{l-2k+1}, * d^* (\Psi \wedge * \alpha^l) \rangle_{L^2}.
\end{aligned}$$

and

$$\begin{aligned}
& \int_M \beta^{l-2k+1} \wedge \Psi \wedge * d \alpha^l \\
& \stackrel{(2.1),(2.3)}{=} (-1)^{l(n-l)+nl+1} \int_M \beta^{l-2k+1} \wedge \Psi \wedge d^* * \alpha^l \\
& \stackrel{(2.1)}{=} (-1)^{n(n-l)} \left\langle \beta^{l-2k+1}, * \left(\Psi \wedge d^* * \alpha^l \right) \right\rangle_{L^2},
\end{aligned}$$

Substituting these into (2.25) yields

$$\begin{aligned}
\langle \mathcal{L}_\Psi \beta^{l-2k+1}, \alpha^l \rangle_{L^2} & = (-1)^{n(n-l)} \left\langle \beta^{l-2k+1}, * \left(d^* (\Psi \wedge * \alpha^l) - \Psi \wedge d^* * \alpha^l \right) \right\rangle_{L^2} \\
& = (-1)^{n(n-l)} \left\langle \beta^{l-2k+1}, - * \mathcal{L}_\Psi * \alpha^l \right\rangle_{L^2},
\end{aligned}$$

from which (2.24) follows. \square

We define the *space of \mathcal{L}_Ψ -harmonic forms* as

$$\begin{aligned}
\mathcal{H}_\Psi^l(M) & := \{ \alpha \in \Omega^l(M) \mid \mathcal{L}_\Psi \alpha = \mathcal{L}_\Psi^* \alpha = 0 \} \\
& \stackrel{(2.24)}{=} \{ \alpha \in \Omega^l(M) \mid \mathcal{L}_\Psi \alpha = \mathcal{L}_\Psi * \alpha = 0 \}
\end{aligned}
\tag{2.26}$$

Evidently, the Hodge-* yields an isomorphism

$$* : \mathcal{H}_\Psi^l(M) \longrightarrow \mathcal{H}_\Psi^{n-l}(M).
\tag{2.27}$$

Since $\mathcal{H}_\Psi^l(M) \subset \ker \mathcal{L}_{\psi;l+2k-1}$ and $\mathcal{H}_\Psi^l(M) \cap \Im(\mathcal{L}_{\psi;l}) = 0$, there is a canonical injection

$$\iota_l : \mathcal{H}_\Psi^l(M) \hookrightarrow H_\psi^l(M).
\tag{2.28}$$

This is analogous to the inclusion of harmonic forms into the deRham cohomology of a manifold, which for a closed manifold is an isomorphism due to the Hodge decomposition (2.23). Therefore, one may hope that the maps ι_l are isomorphism as well. It is not clear if this is always true, but we shall give conditions which assure this to be the case and show that in the applications we have in mind, this condition is satisfied.

Definition 2.6. Let (M, g) be an oriented Riemannian manifold, $\Psi \in \Omega^{2k}(M)$ a parallel form and $\hat{\Psi} := \partial_g \Psi \in \mathcal{MC}^{2k-1}(\Omega^*(M, TM))$ as before.

We say that the differential \mathcal{L}_Ψ is l -regular for $l \in \mathbb{N}$ if there is a direct sum decomposition

$$(2.29) \quad \Omega^l(M) = \ker(\mathcal{L}_{\Psi;l}^*) \oplus \Im(\mathcal{L}_{\Psi;l}).$$

A standard result from ellipticity theory states that \mathcal{L}_Ψ is l -regular if the differential operator $\mathcal{L}_{\Psi;l} : \Omega^{l-2k+1}(M) \rightarrow \Omega^l(M)$ is elliptic, overdetermined elliptic or underdetermined elliptic, see e.g. [Besse1987, p.464, 32 Corollary].

The following theorem now relates the cohomology $H_\Psi^*(M)$ to the \mathcal{L}_Ψ -harmonic forms $\mathcal{H}_\Psi^*(M)$.

Theorem 2.7. *Let (M, g) be an oriented Riemannian manifold, $\Psi \in \Omega^{2k}(M)$ a parallel form and $\hat{\Psi} := \partial_g \Psi \in \mathcal{MC}^{2k-1}(\Omega^*(M, TM))$ as before.*

- (1) *If \mathcal{L}_Ψ is l -regular, then the map ι_l from (2.28) is an isomorphism.*
- (2) *There are direct sum decompositions*

$$(2.30) \quad H_\Psi^l(M) = \mathcal{H}^l(M) \oplus H_\Psi^l(M)_d \oplus H_\Psi^l(M)_{d^*}$$

$$(2.31) \quad \mathcal{H}_\Psi^l(M) = \mathcal{H}^l(M) \oplus \mathcal{H}_\Psi^l(M)_d \oplus \mathcal{H}_\Psi^l(M)_{d^*},$$

where $\mathcal{H}^l(M)$ is the space of harmonic l -forms on M , $H_\Psi^l(M)_d$ and $H_\Psi^l(M)_{d^*}$ are the cohomologies of $(d\Omega^*(M), \mathcal{L}_\Psi)$ and $(d^*\Omega^*(M), \mathcal{L}_\Psi)$, respectively, and where $\mathcal{H}_\Psi^l(M)_d := \mathcal{H}_\Psi^l(M) \cap d\Omega^{l-1}(M)$, $\mathcal{H}_\Psi^l(M)_{d^*} := \mathcal{H}_\Psi^l(M) \cap d^*\Omega^{l+1}(M)$. Moreover, the injective map ι_l from (2.28) preserves this decomposition, i.e.,

$$\iota_l : \mathcal{H}_\Psi^l(M)_d \hookrightarrow H_\Psi^l(M)_d \quad \text{and} \quad \iota_l : \mathcal{H}_\Psi^l(M)_{d^*} \hookrightarrow H_\Psi^l(M)_{d^*}.$$

- (3) *There are isomorphisms*

$$(2.32) \quad d : H_\Psi^l(M)_{d^*} \rightarrow H_\Psi^{l+1}(M)_d \quad \text{and} \quad d^* : H_\Psi^l(M)_d \rightarrow H_\Psi^{l-1}(M)_{d^*}$$

$$(2.33) \quad d : \mathcal{H}_\Psi^l(M)_{d^*} \rightarrow \mathcal{H}_\Psi^{l+1}(M)_d \quad \text{and} \quad d^* : \mathcal{H}_\Psi^l(M)_d \rightarrow \mathcal{H}_\Psi^{l-1}(M)_{d^*}$$

- (4) *If \mathcal{L}_Ψ is $(l+1)$ -regular and $(l-1)$ -regular, then it is also l -regular.*

Proof. Note that (2.29) and $\mathcal{L}_\Psi^2 = 0$ imply that $\ker \mathcal{L}_\Psi|_{\Omega^l(M)} = \mathcal{H}_\Psi^l(M) \oplus \mathcal{L}_\Psi(\Omega^{l-2k+1}(M))$, and from this, the first assertion follows.

If $\alpha_h \in \mathcal{H}^l(M)$ is harmonic, then $d^*\alpha_h = 0$, and $\alpha_h \wedge \Psi \in \mathcal{H}^{l+2k-1}(M)$ by (2.22), so that $d^*(\alpha_h \wedge \Psi) = 0$. This implies that $\mathcal{L}_\Psi(\alpha_h) = 0$, and since $*\alpha_h$ is also harmonic, it follows that $\mathcal{L}_\Psi(*\alpha_h) = 0$, whence $\mathcal{H}^l(M) \subset \mathcal{H}_\Psi^l(M)$.

Furthermore, \mathcal{L}_Ψ commutes with d and d^* up to a sign by (2.20), whence it follows that \mathcal{L}_Ψ preserves the Hodge decomposition (2.23), so that, in particular, the image of \mathcal{L}_Ψ is contained in $d\Omega^*(M) \oplus d^*\Omega^*(M)$. From this, the decompositions (2.30) and (2.31) and hence the second assertion follow.

For the third statement, we first show that

$$(2.34) \quad \begin{aligned} \ker \mathcal{L}_\Psi \cap d\Omega^{l-1}(M) &= d(\ker \mathcal{L}_\Psi \cap d^*\Omega^l(M)), \\ \ker \mathcal{L}_\Psi \cap d^*\Omega^{l+1}(M) &= d^*(\ker \mathcal{L}_\Psi \cap d\Omega^l(M)) \\ \Im(\mathcal{L}_\Psi) \cap d\Omega^{l-1}(M) &= d(\Im \mathcal{L}_\Psi \cap d^*\Omega^l(M)) \\ \Im(\mathcal{L}_\Psi) \cap d^*\Omega^{l+1}(M) &= d^*(\Im \mathcal{L}_\Psi \cap d\Omega^l(M)). \end{aligned}$$

Namely, if $d\alpha^{l-1} \in \ker(\mathcal{L}_\Psi)$ then by the Hodge decomposition we may assume w.l.o.g. that $\alpha^{l-1} = d^*\alpha^l$ with $\alpha^l \in \Omega^l(M)$. Then

$$0 = \mathcal{L}_\Psi(d\alpha^{l-1}) = \mathcal{L}_\Psi(dd^*\alpha^l) \stackrel{(2.20)}{=} dd^*\mathcal{L}_\Psi(\alpha^l),$$

so that $0 = d^*\mathcal{L}_\Psi(\alpha^l) = -\mathcal{L}_\Psi(d^*\alpha^l) = -\mathcal{L}_\Psi(\alpha^{l-1})$. This shows that $\ker \mathcal{L}_\Psi \cap d\Omega^{l-1}(M) \subset d(\ker \mathcal{L}_\Psi \cap d^*\Omega^l(M))$, and the reverse inclusion follows from (2.20). The remaining equalities in (2.34) are shown analogously.

It follows that d and d^* induce the asserted maps in (2.32). Observe that the compositions

$$dd^* : H_\Psi^l(M)_d \rightarrow H_\Psi^l(M)_d \quad \text{and} \quad d^*d : H_\Psi^l(M)_{d^*} \rightarrow H_\Psi^l(M)_{d^*}$$

are induced by the restriction of the Laplacian to $\ker \mathcal{L}_\Psi \cap d\Omega^{l-1}(M)$ and $\ker \mathcal{L}_\Psi \cap d^*\Omega^{l+1}(M)$, respectively, and hence they are isomorphisms. Thus, it follows that the maps in (2.32) are isomorphisms as well.

In addition to the equations in (2.34) it follows from (2.24) that

$$(2.35) \quad \begin{aligned} \ker \mathcal{L}_\Psi^*|_{d\Omega^{l-1}(M)} &= d(\ker \mathcal{L}_\Psi^*|_{d^*\Omega^l(M)}), \quad \text{and} \\ \ker \mathcal{L}_\Psi^*|_{d^*\Omega^{l+1}(M)} &= d^*(\ker \mathcal{L}_\Psi^*|_{d\Omega^l(M)}). \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{H}_\Psi^l(M)_d &= \ker \mathcal{L}_\Psi|_{d\Omega^{l-1}(M)} \cap \ker \mathcal{L}_\Psi^*|_{d\Omega^{l-1}(M)} \\ &\stackrel{(2.34),(2.35)}{=} d(\ker \mathcal{L}_\Psi|_{d^*\Omega^l(M)}) \cap d(\ker \mathcal{L}_\Psi^*|_{d^*\Omega^l(M)}) \\ &\stackrel{(*)}{=} d(\ker \mathcal{L}_\Psi|_{d^*\Omega^l(M)} \cap \ker \mathcal{L}_\Psi^*|_{d^*\Omega^l(M)}) = d\mathcal{H}_\Psi^{l-1}(M)_{d^*}, \end{aligned}$$

where at $(*)$ we used that $d : d^*\Omega^l(M) \rightarrow \Omega^{l-1}(M)$ is injective. Thus, the map $d : \mathcal{H}_\Psi^{l-1}(M)_{d^*} \rightarrow \mathcal{H}_\Psi^l(M)_d$ is an isomorphism, and the same holds for the other map in (2.33), and both are continuous in the Frechét topology.

For the fourth statement, suppose now that \mathcal{L}_Ψ^* is $(l \pm 1)$ -regular. Then by (2.20) and (2.34)

$$\begin{aligned} d\Omega^{l-1}(M) &= d\ker \mathcal{L}_{\Psi;l}^*|_{\Omega^{l-1}(M)} \oplus d\mathcal{L}_{\Psi;l}(\Omega^{l-2k}(M)) \\ &= \ker \mathcal{L}_{\Psi;l}^*|_{d\Omega^{l-1}(M)} \oplus \mathcal{L}_{\Psi;l}(d\Omega^{l-2k}(M)) \\ d^*\Omega^{l+1}(M) &= d^*\ker \mathcal{L}_{\Psi;l}^*|_{\Omega^{l+1}(M)} \oplus d^*\mathcal{L}_{\Psi;l}(\Omega^{l+2-2k}(M)) \\ &= \ker \mathcal{L}_{\Psi;l}^*|_{d^*\Omega^{l+1}(M)} \oplus \mathcal{L}_{\Psi;l}(d^*\Omega^{l+2-2k}(M)), \end{aligned}$$

and since $\mathcal{H}^l(M) \subset \ker \mathcal{L}_\Psi^*$ by the proof of (2), the Hodge decomposition of $\Omega^l(M)$ implies

$$\begin{aligned} \Omega^l(M) &= \mathcal{H}^l(M) \oplus d\Omega^{l-1}(M) \oplus d^*\Omega^{l+1}(M) \\ &= \mathcal{H}^l(M) \oplus \ker \mathcal{L}_{\Psi;l}^*|_{d\Omega^{l-1}(M)} \oplus \mathcal{L}_{\Psi;l}(d\Omega^{l-2k}(M)) \\ &\quad \oplus \ker \mathcal{L}_{\Psi;l}^*|_{d^*\Omega^{l+1}(M)} \oplus \mathcal{L}_{\Psi;l}(d^*\Omega^{l+2-2k}(M)) \\ &= \ker \mathcal{L}_{\Psi;l}^* \oplus \mathfrak{Z}(\mathcal{L}_{\Psi;l}), \end{aligned}$$

where the last equality follows since $\mathcal{L}_{\Psi;l}(\mathcal{H}^{l-2k+1}(M)) = 0$ and $\mathcal{L}_{\Psi;l}^*(\mathcal{H}^l(M)) = 0$. This shows that \mathcal{L}_{Ψ}^* is l -regular. \square

We call a form $\Psi \in \Omega^k(M)$ *multi-symplectic*, if for all $v \in TM$

$$(2.36) \quad \iota_v \Psi = 0 \iff v = 0.$$

Lemma 2.8. *If $\Psi \in \Omega^{2k}(M)$ is multi-symplectic, then the differential operator $\mathcal{L}_{\Psi;l} : \Omega^{l-2k+1}(M) \rightarrow \Omega^l(M)$ is overdetermined elliptic for $l = 2k - 1$ and underdetermined elliptic for $l = n$.*

Proof. By (2.19), the symbol $\sigma_{\xi}(\mathcal{L}_{\Psi,l})$ of $\mathcal{L}_{\Psi} : \Omega^{l-2k+1}(M) \rightarrow \Omega^l(M)$ for $\xi \in T_p^*M$ is given by

$$(2.37) \quad \begin{aligned} \sigma_{\xi}(\mathcal{L}_{\Psi,l}) : \Lambda^{l-2k+1}T_p^*M &\longrightarrow \Lambda^l T_p^*M, \\ \alpha^{l-2k+1} &\longmapsto (\iota_{\xi\#}\Psi) \wedge \alpha^{l+2k-1}. \end{aligned}$$

If $l = 2k - 1$ or $l = n$ then either the domain or the range of $\sigma_{\xi}(\mathcal{L}_{\Psi,l})$ is one dimensional, whence the injectivity or surjectivity of $\sigma_{\xi}(\mathcal{L}_{\Psi,l})$ is given unless $\sigma_{\xi}(\mathcal{L}_{\Psi,l}) = 0$.

For $l = 2k - 1$, observe that $\sigma_{\xi}(\mathcal{L}_{\Psi,l})(1) = (\iota_{\xi\#}\Psi) \neq 0$ for $\xi \neq 0$ as Ψ is multi-symplectic; for $l = n$,

$$\sigma_{\xi}(\mathcal{L}_{\Psi,n})(*\iota_{\xi\#}\Psi) = (\iota_{\xi\#}\Psi) \wedge (*\iota_{\xi\#}\Psi) = |\iota_{\xi\#}\Psi|^2 \text{vol}_p,$$

which again is non-zero for $\xi \neq 0$ as Ψ is multi-symplectic. \square

As an immediate consequence of this and Theorem 2.7, we obtain the

Corollary 2.9. *Let (M, g) and $\Psi \in \Omega^{2k}(M)$ be a parallel multi-symplectic form. Then*

$$H_{\Psi}^{2k-1}(\Omega^*(M)) = \mathcal{H}_{\Psi}^{2k-1}(M).$$

We also can make some statement for $H_{\Psi}^l(M)$ for special values of l .

Proposition 2.10. *Let (M, g) and $\Psi \in \Omega^{2k}(M)$ be as before. Then*

$$H_{\Psi}^0(\Omega^*(M)) \cong \mathcal{H}_{\Psi}^0(M) = \{f \in C^{\infty}(M) \mid \iota_{df\#}\Psi = 0\},$$

If Ψ is multi-symplectic, then $\mathcal{H}_{\Psi}^0(M) = \mathcal{H}^0(M)$ and $H_{\Psi}^n(\Omega^(M)) \cong \mathcal{H}_{\Psi}^n(M) = \mathcal{H}^n(M)$.*

Proof. Let $f \in \Omega^0(M) = C^{\infty}(M)$. Then by (2.19), we have

$$\mathcal{L}_{\Psi}(f) = \iota_{df\#}\Psi,$$

which implies the statement for $H_{\Psi}^0(\Omega^*(M))$.

If Ψ is multisymplectic, then $\iota_{df\#}\Psi = 0$ iff $df = 0$, showing that $\mathcal{H}_{\Psi}^0(M) = \mathcal{H}^0(M)$. Moreover, using Lemma 2.8 for $l = n$, we see from Theorem 2.7(1) that $H_{\Psi}^n(\Omega^*(M)) = \mathcal{H}_{\Psi}^n(M)$, and the latter space equals $*\mathcal{H}_{\Psi}^0(M)$ by (2.27), which by the above equals $*\mathcal{H}^0(M) = \mathcal{H}^n(M)$. \square

Observe that $H_{\Psi}^0(\Omega^*(M))$ is infinite dimensional if Ψ is not multi-symplectic.

Proposition 2.11. *Let (M, g) and $\Psi \in \Omega^{2k}(M)$ be as above, and suppose that Ψ is multi-symplectic. Then*

$$\ker(\mathcal{L}_{\Psi;2k}) = \{\alpha \in \Omega^1(M) \mid \mathcal{L}_{\alpha\#}(*\Psi) = 0 \quad \text{and} \quad d^*\alpha = 0\}.$$

In particular, if $k \geq 2$ then $\ker \mathcal{L}_{\Psi;2k} = \mathcal{H}_{\Psi}^1(M) \cong H_{\Psi}^1(M)$ and

$$\mathcal{H}_{\Psi}^{n-1}(M) = \{\alpha \in \Omega^{n-1}(M) \mid \mathcal{L}_{(*\alpha)\#}(*\Psi) = 0 \quad \text{and} \quad d\alpha = 0\}.$$

Proof. We calculate for $\alpha \in \Omega^1(M)$

$$\begin{aligned} \mathcal{L}_{\Psi}(\alpha) &\stackrel{(2.18)}{=} (d^*\alpha) \cdot \Psi - d^*(\alpha \wedge \Psi) = (d^*\alpha) \cdot \Psi + *d*(\alpha \wedge \Psi) \\ &\stackrel{(2.2)}{=} (d^*\alpha) \cdot \Psi + *d(\iota_{\alpha\#} * \Psi) \\ &\stackrel{(2.1)}{=} *((d^*\alpha) \cdot * \Psi + \mathcal{L}_{\alpha\#}(*\Psi)), \end{aligned}$$

using again $d^*\Psi = 0$ and hence, $d(\iota_{\alpha\#} * \Psi) = \mathcal{L}_{\alpha\#}(*\Psi)$ in the last step.

Thus, $\mathcal{L}_{\Psi}(\alpha) = 0$ iff $\mathcal{L}_{\alpha\#}(*\Psi) = -(d^*\alpha) \cdot * \Psi$. If this is the case, taking the exterior derivative implies that

$$\begin{aligned} -dd^*\alpha \wedge * \Psi &\stackrel{d^*\Psi=0}{=} -d((d^*\alpha) \cdot * \Psi) \\ &= d\mathcal{L}_{\alpha\#}(*\Psi) = \mathcal{L}_{\alpha\#}(d * \Psi) = 0, \end{aligned}$$

whence

$$0 = -dd^*\alpha \wedge * \Psi \stackrel{(2.2)}{=} *\iota_{(dd^*\alpha)\#} \Psi,$$

so that $\iota_{(dd^*\alpha)\#} \Psi = 0$ and hence, by (2.36), $dd^*\alpha = 0$. This implies that $d^*\alpha = 0$ and hence, $\mathcal{L}_{\alpha\#}(*\Psi) = 0$, showing the statement on $\ker \mathcal{L}_{\Psi}|_{\Omega^1(M)}$.

The last statement then follows from the definition and (2.27). \square

3. THE FRÖLICHER-NIJENHUIS COHOMOLOGY OF G_2 -MANIFOLDS

In this section we shall apply the cohomology definition from the preceding section to the parallel 4-form in a G_2 -manifold. We first collect some basic facts on the representation of the exceptional group G_2 , see e.g. [Humphreys1978].

Let V be an oriented 7-dimensional vector space. A G_2 -structure on V is a form $\varphi \in \Lambda^3 V^*$ for which there is a positively oriented basis (e_i) of V with the dual basis (e^i) of V^* such that

$$(3.1) \quad \varphi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},$$

where $e^{i_1 \dots i_k}$ is short for $e^{i_1} \wedge \dots \wedge e^{i_k}$. Setting $\text{vol} := e^{1 \dots 7}$, φ uniquely determines an inner product g_{φ} by the identity

$$(3.2) \quad g_{\varphi}(u, v) \text{ vol} = \frac{1}{6}((\iota_u \varphi) \wedge (\iota_v \varphi) \wedge \varphi),$$

and it follows that any oriented basis (e_i) for which (3.1) holds is orthonormal w.r.t. g_{φ} . Thus, the Hodge-dual of φ w.r.t. g_{φ} is given by

$$(3.3) \quad *\varphi = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}.$$

The stabilizer of φ is known to be the exceptional 14-dimensional simple Lie group $G_2 \subset \text{Gl}(V)$, and the elements of G_2 preserve both g_φ and vol , i.e., $G_2 \subset \text{SO}(V, g_\varphi)$.

We summarize important known facts about the decomposition of exterior powers of G_2 -modules into irreducible summands which are well known, see e.g. [Kar2005, Section 2]. We denote by V_k the k -dimensional irreducible G_2 -module if there is a unique such module. For instance, V_7 is the irreducible 7-dimensional G_2 -module from above, and $V_7^* \cong V_7$. For its exterior powers, we obtain the decompositions

$$(3.4) \quad \begin{aligned} \Lambda^0 V_7 &\cong \Lambda^7 V_7 \cong V_1, & \Lambda^2 V_7 &\cong \Lambda^5 V_7 \cong V_7 \oplus V_{14}, \\ \Lambda^1 V_7 &\cong \Lambda^6 V_7 \cong V_7, & \Lambda^3 V_7 &\cong \Lambda^4 V_7 \cong V_1 \oplus V_7 \oplus V_{27}, \end{aligned}$$

where $\Lambda^k V_7 \cong \Lambda^{7-k} V_7$ due to G_2 -invariance of the Hodge isomorphism $*$: $\Lambda^k V_7 \rightarrow \Lambda^{7-k} V_7^* \cong \Lambda^{7-k} V_7$. We denote by $\Lambda_l^k V_7 \subset \Lambda^k V_7$ the subspace isomorphic to V_l in the above notation. Evidently, $\Lambda_1^3 V_7$ and $\Lambda_1^4 V_7$ are spanned by φ and $*\varphi$, respectively. For the remaining spaces in the decompositions of $\Lambda^k V_7$ we obtain the following descriptions.

$$(3.5) \quad \begin{aligned} \Lambda_7^2 V_7 &= \{\iota_v \varphi \mid v \in V_7\}, \\ \Lambda_{14}^2 V_7 &= \{\alpha^2 \in \Lambda^2 V_7 \mid \alpha^2 \wedge *\varphi = 0\}, \\ \Lambda_7^3 V_7 &= \{\iota_v *\varphi \mid v \in V_7\} = \{*(v^\flat \wedge \varphi) \mid v \in V_7\}, \\ \Lambda_{27}^3 V_7 &= \{\alpha^3 \in \Lambda^3 V_7 \mid \alpha^3 \wedge \varphi = \alpha^3 \wedge *\varphi = 0\}, \\ \Lambda_7^4 V_7 &= V_7 \wedge \varphi = \{v \wedge \varphi \mid v \in V_7\}, \\ \Lambda_{27}^4 V_7 &= \{\alpha^4 \in \Lambda^4 V_7 \mid \alpha^4 \wedge \varphi = *\alpha^4 \wedge \varphi = 0\}, \\ \Lambda_7^5 V_7 &= V_7 \wedge *\varphi = \{v \wedge *\varphi \mid v \in V_7\}, \\ \Lambda_{14}^5 V_7 &= \{\alpha^5 \in \Lambda^5 V_7 \mid \alpha^5 \wedge (\iota_v \varphi) = 0 \text{ for all } v \in V_7\}. \end{aligned}$$

A G_2 -manifold is a pair (M, φ) consisting of a 7-dimensional oriented manifold M with a parallel 3-form φ such that at each $p \in M$ there is an oriented basis (e_i) of $T_p M$ with dual basis (e^i) of $T_p^* M$ such that φ_p has the form (3.1). Then φ induces a Riemannian metric g_φ on M , and (3.4) induces a decomposition of differential forms and canonical projections by

$$(3.6) \quad \Omega_l^k(M) := \Gamma(M, \Lambda_l^k T^* M), \quad \pi_l^k : \Omega^k(M) \longrightarrow \Omega_l^k(M).$$

Moreover, we let

$$(3.7) \quad \begin{aligned} \Omega_l^{k;d}(M) &:= \Omega_l^k(M) \cap d\Omega^{k-1}(M), \\ \Omega_l^{k;d^*}(M) &:= \Omega_l^k(M) \cap d^*\Omega^{k+1}(M) \end{aligned}$$

It is our aim to investigate these spaces for a G_2 -manifold (M, φ) . First of all, the Hodge- $*$ gives isomorphisms

$$(3.8) \quad \Omega_l^{k;d}(M) \xleftarrow{*} \Omega_l^{7-k;d^*}(M)$$

Since the decomposition of $\Lambda^k T^*M$ into its irreducible components is preserved by parallel translation, it follows from the Weitzenböck formula (see e.g. [Besse1987, (1.154)]) that their sections are preserved by the Laplace operator, and since this operator also commutes with d and d^* , we have

$$(3.9) \quad \Delta \Omega_l^{k;d}(M) = \Omega_l^{k;d}(M) \quad \text{and} \quad \Delta \Omega_l^{k;d^*}(M) = \Omega_l^{k;d^*}(M)$$

as well as

$$(3.10) \quad \Delta \pi_l^k(\alpha^k) = \pi_l^k(\Delta \alpha^k) \quad \text{for all } \alpha^k \in \Omega^k(M).$$

Definition 3.1. For $\beta^2 \in d\Omega^1(M)$, we define the 4-form

$$(3.11) \quad \alpha_{\beta^2}^4 := (d^* \beta^2) \wedge \varphi - d^*(\beta^2 \wedge \varphi).$$

Moreover, define

$$(3.12) \quad \begin{aligned} V_3^{d^*}(M) &:= \{*\alpha_{\beta^2}^4 \mid \beta^2 \in d\Omega^1(M)\} \quad \text{and} \\ V_4^d(M) &:= \{\alpha_{\beta^2}^4 \mid \beta^2 \in d\Omega^1(M)\}. \end{aligned}$$

Lemma 3.2. For every $\beta^2 \in d\Omega^1(M)$, we have $\alpha_{\beta^2}^4 \in d\Omega^3(M) \cap (\Omega_{27}^{4;d}(M))^\perp$. Furthermore,

$$(3.13) \quad \alpha_{\beta^2}^4 \wedge \varphi = 0 \quad \text{and} \quad *\alpha_{\beta^2}^4 \wedge \varphi = -2d*\beta^2 \in d\Omega^5(M).$$

Proof. First observe that

$$d\alpha_{\beta^2}^4 = (dd^*\beta^2) \wedge \varphi - dd^*(\beta^2 \wedge \varphi) \stackrel{d\beta^2=0}{=} (\Delta\beta^2) \wedge \varphi - \Delta(\beta^2 \wedge \varphi) \stackrel{(2.22)}{=} 0.$$

Note that

$$\alpha_{\Delta\beta^2}^4 = (\Delta d^*\beta^2) \wedge \varphi - d^*((\Delta\beta^2) \wedge \varphi) \stackrel{(2.22)}{=} \Delta\alpha_{\beta^2}^4,$$

and since $d\Omega^1(M) = \Delta d\Omega^1(M)$, it follows that $\alpha_{\beta^2}^4 \in \Delta\Omega^4(M) = \mathcal{H}^4(M)^\perp$, and this together with $d\alpha_{\beta^2}^4 = 0$ implies that $\alpha_{\beta^2}^4 \in d\Omega^3(M)$. If $\gamma^4 \in \Omega_{27}^{4;d}(M)$, then

$$\langle \alpha_{\beta^2}^4, \gamma^4 \rangle_{L^2} = \int_M d^*\beta^2 \wedge \underbrace{\varphi \wedge *\gamma^4}_{=0} - \langle \beta^2 \wedge \varphi, \underbrace{d\gamma^4}_{=0} \rangle_{L^2} = 0,$$

so that $\alpha_{\beta^2}^4 \in (\Omega_{27}^{4;d}(M))^\perp$. Furthermore,

$$\begin{aligned} \alpha_{\beta^2}^4 \wedge \varphi &= (d^*\beta^2) \wedge \underbrace{\varphi \wedge \varphi}_{=0} + (*d*(\beta^2 \wedge \varphi)) \wedge \varphi = (d*(\beta^2 \wedge \varphi)) \wedge *\varphi \\ &= d(*(\beta^2 \wedge \varphi) \wedge *\varphi) \stackrel{(3.5)}{=} d(\pi_7^2(*(\beta^2 \wedge \varphi)) \wedge *\varphi) \\ &\stackrel{\text{Lemma 7.2}}{=} 2d(\pi_7^2(\beta^2) \wedge *\varphi) \stackrel{(3.5)}{=} 2d(\beta^2 \wedge *\varphi) \stackrel{d\beta^2=0}{=} 0, \end{aligned}$$

and

$$\begin{aligned}
\alpha_{\beta^2}^4 \wedge \varphi &= ((d^*\beta^2 \wedge \varphi)) \wedge \varphi + (d*(\beta^2 \wedge \varphi)) \wedge \varphi \\
&\stackrel{(7.2), \text{Lemma 7.2}}{=} -4*d*\beta^2 + d((2\pi_7^2(\beta^2) - \pi_{14}^2(\beta^2)) \wedge \varphi) \\
&\stackrel{\text{Lemma 7.2}}{=} -4d*\beta^2 + d(4*\pi_7^2(\beta^2) + *\pi_{14}^2(\beta^2)) \\
&= -2d*\beta^2 + d*((2\pi_7^2(\beta^2) - \pi_{14}^2(\beta^2))) \\
&\stackrel{\text{Lemma 7.2}}{=} -2d*\beta^2 + d*(\beta^2 \wedge \varphi) = -2d*\beta^2. \quad \square
\end{aligned}$$

Proposition 3.3. *For a closed G_2 -manifold (M, φ) and the spaces defined above, we have the following:*

- (1) $\Omega_7^{2;d}(M) = \Omega_{14}^{2;d}(M) = 0$ and $\Omega_7^{5;d^*}(M) = \Omega_{14}^{5;d^*}(M) = 0$.
- (2) $\Omega_1^{3;d}(M) = \Omega_1^{3;d^*}(M) = 0$ and $\Omega_1^{4;d}(M) = \Omega_1^{4;d^*}(M) = 0$.
- (3) $\Omega_7^{3;d}(M) = \Omega_7^{4;d^*}(M) = 0$.
- (4) *We have the following commutative diagrams of isomorphisms:*

(3.14)

$$\begin{array}{ccc}
\Omega_7^{2;d^*}(M) & \xrightleftharpoons[d^*]{d} & dd^*(\Omega_1^3(M)) \\
\Downarrow * & & \Downarrow * \\
\Omega_7^{5;d}(M) & \xrightleftharpoons[d]{d^*} & d^*d(\Omega_1^4(M))
\end{array}
\quad
\begin{array}{ccc}
\Omega_7^{3;d^*}(M) & \xrightleftharpoons[d^*]{d} & dd^*\Omega_1^4(M) \\
\Downarrow * & & \Downarrow * \\
\Omega_7^{4;d}(M) & \xrightleftharpoons[d]{d^*} & d^*d\Omega_1^3(M)
\end{array}$$

$$\begin{array}{ccc}
\Omega_{14}^{2;d^*}(M) & \xrightleftharpoons[d^*]{d} & \Omega_{27}^{3;d}(M) \\
\Downarrow * & & \Downarrow * \\
\Omega_{14}^{5;d}(M) & \xrightleftharpoons[d]{d^*} & \Omega_{27}^{4;d^*}(M)
\end{array}
\quad
\begin{array}{c}
\Omega_{27}^{3;d^*}(M) \oplus^\perp V_3^{d^*}(M) \\
d^* \left(\Downarrow * \right) d \\
\Omega_{27}^{4;d}(M) \oplus^\perp V_4^d(M)
\end{array}$$

In fact, for the spaces in these diagrams we have the descriptions

$$(3.15) \quad \Omega_7^{2;d^*}(M) = \{\iota_{(df)\#} \varphi \mid f \in C^\infty(M)\} = d^*(\Omega_1^3(M)),$$

$$(3.16) \quad \Omega_7^{3;d^*}(M) = \{\iota_{(df)\#} *\varphi \mid f \in C^\infty(M)\} = d^*(\Omega_1^4(M)),$$

and

$$(3.17) \quad \Omega_{27}^{3;d^*}(M) \oplus^\perp V_3^{d^*}(M) = \{\alpha^3 \in d^*\Omega^4(M) \mid \alpha^3 \wedge *\varphi = 0, d(\alpha^3 \wedge \varphi) = 0\}.$$

Proof. Let $\alpha^2 \in \Omega_l^2(M)$, where $l = 7, 14$. Then according to Lemma 7.2 there are constants $c_l \in \mathbb{R}$ such that $*\alpha^2 = c_l(\alpha^2 \wedge \varphi)$. Thus, for $\alpha^2 \in \Omega_l^2(M)$ we have

$$(3.18) \quad d^*\alpha^2 \stackrel{(2.3)}{=} *d*\alpha^2 = c_l*d(\alpha^2 \wedge \varphi) = c_l*(d\alpha^2 \wedge \varphi).$$

Therefore, if $\alpha^2 \in \Omega_l^{2;d}(M)$, then (3.18) implies that $d^*\alpha = 0$, whence α^2 is both exact and coclosed and hence $\alpha^2 = 0$, showing that $\Omega_l^{2;d}(M) = 0$ and therefore, $\Omega_l^{5;d^*}(M) = 0$ by (3.8). This shows the first statement.

The second statement is immediate as $d(f\varphi) = df \wedge \varphi$ and $d(f*\varphi) = df \wedge *\varphi$ vanish iff $df = 0$ and hence $f\varphi$ and $f*\varphi$ are harmonic. Thus, $\Omega_1^{k;d}(M) = 0$ for $k = 3, 4$, and the other two follow from (3.8).

For the third statement, let $\iota_{(\alpha^1)^\#}*\varphi \in \Omega_7^{3;d}(M)$. Then $0 = d(\iota_{(\alpha^1)^\#}*\varphi) = \mathcal{L}_{(\alpha^1)^\#}*\varphi$, which implies that $(\alpha^1)^\#$ is a Killing vector field. As G_2 -manifolds are Ricci-flat, Bochner's theorem implies that $(\alpha^1)^\#$ is parallel and hence, so is $\iota_{(\alpha^1)^\#}*\varphi$. Thus, $\iota_{(\alpha^1)^\#}*\varphi \in \Omega_7^{3;d}(M)$ is harmonic, and since it is also exact, it must vanish. That is, $\Omega_7^{3;d}(M) = 0$, and by (3.8) $\Omega_7^{4;d^*}(M) = 0$ as well.

For the last part, we shall first prove (3.15). Namely, for $f \in C^\infty(M)$

$$\iota_{df\#\varphi} \stackrel{(2.1),(2.2)}{=} *(df \wedge *\varphi) = *d(*f\varphi) \stackrel{(2.3)}{=} -d^*(f\varphi) \in d^*\Omega_1^3(M).$$

It follows that $\iota_{df\#\varphi} \in d^*\Omega_1^3(M) \subset \Omega_7^{2;d^*}(M)$. Conversely, if $\alpha^2 = \iota_{(\alpha^1)^\#}\varphi \in \Omega_7^{2;d^*}(M)$, then

$$0 = d^*\alpha^2 \stackrel{(2.3)}{=} *d*\iota_{(\alpha^1)^\#}\varphi \stackrel{(2.2)}{=} *d(\alpha^1 \wedge *\varphi) = *(d\alpha^1 \wedge *\varphi),$$

which implies that $d\alpha^1 \in \Omega_{14}^{2;d}(M) = 0$, i.e., $\alpha^1 = \alpha_h^1 + df$ for some $\alpha_h^1 \in \mathcal{H}^1(M)$ and $f \in C^\infty(M)$, i.e., so that $\alpha^2 = \iota_{df\#\varphi} + \iota_{(\alpha_h^1)^\#}\varphi$. Since we already showed that $\iota_{df\#\varphi} \in \Omega_7^{2;d^*}(M)$ and because $\iota_{(\alpha_h^1)^\#}\varphi$ is harmonic, it follows that $\alpha_h^1 = 0$, and hence, $\alpha^2 \in \Omega_7^{2;d^*}(M)$ iff $\alpha^2 = \iota_{df\#\varphi} = -d^*(f\varphi)$ for some $f \in C^\infty(M)$ which shows (3.15), so that $d\Omega_7^{2;d^*}(M) = dd^*(\Omega_1^3(M))$ and $d^*dd^*(\Omega_1^3(M)) = \Delta\Omega_7^{2;d^*}(M) = \Omega_7^{2;d^*}(M)$.

The second diagram in (3.14) and (3.16) is dealt with in a similar fashion; in fact, by replacing $\alpha^2 = \iota_{(\alpha^1)^\#}\varphi$ by $\alpha^3 = \iota_{(\alpha^1)^\#}*\varphi$, we show by literally the same proof that (3.16) holds, whence the second diagram commutes as well.

Now let us consider the third diagram in (3.14). If $\alpha^2 \in \Omega_{14}^{2;d^*}(M)$, then $d\alpha^2 \wedge *\varphi = 0$ and (3.18) implies that $d\alpha^2 \wedge \varphi = 0$, which shows that

$$(3.19) \quad d\Omega_{14}^{2;d^*}(M) \subset \Omega_{27}^{3;d}(M).$$

Conversely, suppose that $\alpha^3 \in \Omega_{27}^{3;d}(M)$, and let $\alpha^2 := d^*\alpha^3$. Since $\alpha^2 \in d^*\Omega^3(M) \subset \Delta\Omega^2(M)$, it follows from (2.22) that $*(\alpha^2 \wedge \varphi) \in \Delta\Omega^2(M) = (\mathcal{H}^2(M))^\perp$. Moreover,

$$(3.20) \quad d^*(\alpha^2 \wedge \varphi) \stackrel{(2.3),(2.1)}{=} *(d\alpha^2 \wedge \varphi) = *(\Delta\alpha^3 \wedge \varphi) \stackrel{(2.22)}{=} *\Delta(\alpha^3 \wedge \varphi) = 0.$$

Thus, $*(\alpha^2 \wedge \varphi) \in d^*\Omega^3(M)$ and therefore,

$$3\pi_7^2(\alpha^2) \stackrel{\text{Lemma 7.2}}{=} \alpha^2 + *(\alpha^2 \wedge \varphi) \in \Omega_7^{2;d^*}(M),$$

so that by (3.15) $\pi_7^2(\alpha^2) = \iota_{(df)^\#} \varphi$ for some $f \in C^\infty(M)$. But then,

$$\alpha^2 \wedge * \varphi = \pi_7^2(\alpha^2) \wedge * \varphi = \iota_{(df)^\#} \varphi \wedge * \varphi \stackrel{(7.1)}{=} 3 * df,$$

whence,

$$3d * df = d\alpha^2 \wedge * \varphi = \Delta \alpha^3 \wedge * \varphi = 0$$

as $\alpha^3 \in \Omega_{27}^3(M)$, and hence $d^*df = 0$, so that $df = 0$ and hence, $\pi_7^2(\alpha^2) = 0$ which implies that $\alpha^2 \in \Omega_{14}^{2;d^*}(M)$.

Thus, we have shown that $d^*\Omega_{27}^{3;d}(M) \subset \Omega_{14}^{2;d^*}(M)$, which together with (3.19) shows that the maps in the third diagram are isomorphisms.

We now show (3.17). The inclusion \subset is obvious from the definition of $\Omega_{27}^{3;d^*}(M)$ and Lemma 3.2. For the converse, let $\alpha^3 \in d^*\Omega^4(M)$ be such that $\alpha^3 \wedge * \varphi = 0$ and $d\alpha^3 \wedge \varphi = 0$. Let $\tilde{\alpha}^3 \in d^*\Omega^4(M)$ be such that $\alpha^3 = \Delta \tilde{\alpha}^3$, let $\beta^2 := d * (\tilde{\alpha}^3 \wedge \varphi) \in d\Omega^1(M)$, and

$$\hat{\alpha}^3 := \alpha^3 + \frac{1}{2} * \alpha_{\beta^2}^4$$

with $\alpha_{\beta^2}^4$ from (3.11). Since $\alpha_{\beta^2}^4 \in d\Omega^3(M)$ by Lemma 3.2, it follows that $\hat{\alpha}^3 \in d^*\Omega^4(M)$. Moreover,

$$\hat{\alpha}^3 \wedge * \varphi = \alpha^3 \wedge * \varphi + \frac{1}{2} * \alpha_{\beta^2}^4 \wedge * \varphi = 0 + \frac{1}{2} \alpha_{\beta^2}^4 \wedge \varphi \stackrel{(3.13)}{=} 0,$$

and

$$\begin{aligned} \hat{\alpha}^3 \wedge \varphi &= \alpha^3 \wedge \varphi + \frac{1}{2} (*\alpha_{\beta^2}^4) \wedge \varphi \stackrel{(3.13)}{=} \alpha^3 \wedge \varphi - d * \beta^2 \\ &= \alpha^3 \wedge \varphi - dd^*(\tilde{\alpha}^3 \wedge \varphi) = \alpha^3 \wedge \varphi - \Delta(\tilde{\alpha}^3 \wedge \varphi) + d^*d(\tilde{\alpha}^3 \wedge \varphi) \\ &\stackrel{(2.22)}{=} \alpha^3 \wedge \varphi - (\Delta \tilde{\alpha}^3) \wedge \varphi + d^*d(\tilde{\alpha}^3 \wedge \varphi) = d^*d(\tilde{\alpha}^3 \wedge \varphi). \end{aligned}$$

Now

$$\Delta d(\tilde{\alpha}^3 \wedge \varphi) \stackrel{(2.22)}{=} d((\Delta \tilde{\alpha}^3) \wedge \varphi) = d(\alpha^3 \wedge \varphi) = 0,$$

so that $d(\tilde{\alpha}^3 \wedge \varphi) \in \mathcal{H}^7(M) \cap d\Omega^6(M) = 0$. Thus, $d(\tilde{\alpha}^3 \wedge \varphi) = 0$, whence $\hat{\alpha}^3 \wedge \varphi = 0$.

All of this now implies that $\hat{\alpha}^3 \in \Omega_{27}^{3;d^*}(M)$ and therefore,

$$\alpha^3 = \hat{\alpha}^3 - \frac{1}{2} * \alpha_{\beta^2}^4 \in \Omega_{27}^{3;d^*}(M) \oplus V_3^{d^*}(M),$$

which shows (3.17). Thus, in order to show that the maps in the last diagram in (3.14) are isomorphisms, we have to show that $*d$ preserves $\Omega_{27}^{3;d^*}(M) \oplus$

$V_3^{d^*}(M)$. If α^3 is an element of this space, then evidently, $*d\alpha^3 = d^* * \alpha^3 \in d^* \Omega^4(M)$. Moreover,

$$*d\alpha^3 \wedge *\varphi = d\alpha^3 \wedge \varphi = 0$$

$$(d * d\alpha^3) \wedge \varphi = (*d * d\alpha^3) \wedge *\varphi \stackrel{d^* \alpha^3 = 0}{=} (\Delta \alpha^3) \wedge *\varphi \stackrel{(2.22)}{=} \Delta(\alpha^3 \wedge *\varphi) = 0.$$

Thus, $*d$ maps $\Omega_{27}^{3;d^*}(M) \oplus V_3^{d^*}(M)$ to itself, and since the restriction of $(*d)^2$ to $\Omega_{27}^{3;d^*}(M) \oplus V_3^{d^*}(M)$ coincides with the Laplacian and hence is an isomorphism, it follows that $*d : \Omega_{27}^{3;d^*}(M) \oplus V_3^{d^*}(M) \rightarrow \Omega_{27}^{3;d^*}(M) \oplus V_3^{d^*}(M)$ is an isomorphism as well. \square

Proposition 3.4. *For any G_2 -manifold, all spaces in the diagrams (3.14) are infinite dimensional.*

Proof. By (3.15), (3.16) and (3.13), there are isomorphisms

$$\begin{aligned} dC^\infty(M) &\xrightarrow{\cong} \Omega_7^{2;d^*}(M) & df &\longmapsto \iota_{(df)^\#} \varphi \\ dC^\infty(M) &\xrightarrow{\cong} \Omega_7^{3;d^*}(M) & df &\longmapsto \iota_{(df)^\#} * \varphi \\ d\Omega^1(M) &\xrightarrow{\cong} V_3^{d^*}(M) & \beta^2 &\longmapsto *\alpha_{\beta^2}^4, \end{aligned}$$

showing that these spaces are infinite dimensional. In order to show that $\Omega_{14}^{2;d^*}(M)$ and $\Omega_{27}^{3;d^*}(M)$ are infinite dimensional, we assert that there are direct sum decompositions

$$(3.21) \quad \Omega_{14}^2(M) = \mathcal{H}_{14}^2(M) \oplus \Omega_{14}^{2;d^*}(M) \oplus \pi_{14}^2(d\Omega^1(M)),$$

$$(3.22) \quad \Omega_{27}^3(M) = \mathcal{H}_{27}^3(M) \oplus \Omega_{27}^{3;d^*}(M) \oplus \pi_{27}^3(d\Omega^2(M)).$$

If these decompositions hold, then we apply Lemma 7.4 to the differential operators $\Omega^1(M) \ni \alpha^1 \mapsto \pi_{14}^2(d\alpha^1) \in \Omega_{14}^2(M)$ and $\Omega^2(M) \ni \alpha^2 \mapsto \pi_{27}^3(d\alpha^2) \in \Omega_{27}^3(M)$, respectively, to conclude that $\Omega_{14}^{2;d^*}(M)$ and $\Omega_{27}^{3;d^*}(M)$ are infinite dimensional, as the space of harmonic forms is finite dimensional.

To see the decompositions (3.21) and (3.22), we consider the differential operators $\phi_1 : \Omega^1(M) \rightarrow \Omega_{14}^2(M) \oplus C^\infty(M)$ and $\phi_2 : \Omega^2(M) \rightarrow \Omega_{27}^3(M) \oplus \Omega^1(M)$ given by

$$\phi_1(\alpha^1) := \pi_{14}^2(d\alpha^1) + d^* \alpha^1, \quad \phi_2(\alpha^2) := \pi_{27}^3(d\alpha^2) + d^* \alpha^2.$$

Their formal adjoints are given by $\phi_1^* : \Omega_{14}^2(M) \oplus C^\infty(M) \rightarrow \Omega^1(M)$ and $\phi_2^* : \Omega_{27}^3(M) \oplus \Omega^1(M) \rightarrow \Omega^2(M)$ as

$$\phi_1^*(\beta_{14}^2, f) = d^* \beta_{14}^2 + df \quad \text{and} \quad \phi_2^*(\beta_{27}^3, \beta^1) = d^* \beta_{27}^3 + d\beta^1.$$

By the Hodge decomposition $(\beta_{14}^2, f) \in \ker \phi_1^*$ iff $d^* \beta_{14}^2 = 0$ and $df = 0$. In this case, since there is an orthogonal decompositions $\mathcal{H}^2(M) = \mathcal{H}_7^2(M) \oplus \mathcal{H}_{14}^2(M)$, it follows that the harmonic part of β_{14}^2 must be an element of $\mathcal{H}_{14}^2(M)$, whence the coexact part must be an element of $\Omega_{14}^2(M)$ as well. Thus, it follows that

$$\ker \phi_1^* = (\mathcal{H}_{14}^2(M) \oplus \Omega_{14}^{2;d^*}(M)) \oplus \mathcal{H}^0(M),$$

and analogously,

$$\ker \phi_2^* = (\mathcal{H}_{27}^3(M) \oplus \Omega_{27}^{3;d^*}(M)) \oplus \ker d|_{\Omega^1(M)}.$$

Thus, if we can show that ϕ_1, ϕ_2 are overdetermined elliptic, then $\Omega_{14}^2(M) \oplus C^\infty(M) = \phi_1(\Omega^1(M)) \oplus \ker \phi_1^*$ and $\Omega_{27}^3(M) \oplus \Omega^1(M) = \phi_2(\Omega^2(M)) \oplus \ker \phi_2^*$, and from this, (3.21) and (3.22) follows.

In order to show that ϕ_1 and ϕ_2 are overdetermined elliptic, we calculate their symbols for $0 \neq \xi \in T_p^*M$ as

$$\sigma_\xi^1(\alpha^1) = \pi_{14}^2(\xi \wedge \alpha^1) - \iota_{\xi^\#} \alpha^1, \quad \sigma_\xi^2(\alpha^2) = \pi_{27}^3(\xi \wedge \alpha^2) - \iota_{\xi^\#} \alpha^2.$$

Let $\alpha^1 \in \ker \sigma_\xi^1$. Then $\xi \wedge \alpha^1 = \iota_v \varphi \in \Lambda_7^2 T_p^*M$ for some $v \in T_pM$. But then, (3.2) implies that

$$6\|v\|^2 \text{ vol} = (\xi \wedge \alpha^1)^2 \wedge \varphi = 0 \Rightarrow v = 0 \Rightarrow \xi \wedge \alpha^1 = 0,$$

so that $\alpha^1 = c\xi$ for some $c \in \mathbb{R}$. But then, $0 = \iota_{\xi^\#} \alpha^1 = c\|\xi\|^2$ and $\xi \neq 0$ implies that $c = 0$ and hence, $\alpha^1 = 0$, so that $\ker \sigma_\xi^1 = 0$.

Similarly, $\alpha^2 \in \ker \sigma_\xi^2$ iff $\iota_{\xi^\#} \alpha^2 = 0$ and $\xi \wedge \alpha^2 \in \Lambda_7^3 T_p^*M \oplus \Lambda_1^3 T_p^*M$. The second equation implies that there is a $c \in \mathbb{R}$ and $v \in T_pM$ such that

$$\xi \wedge \alpha^2 = c\varphi + \iota_v * \varphi \Rightarrow 0 = \xi \wedge (c\varphi + \iota_v * \varphi) \wedge \varphi \stackrel{(7.2)}{=} 4\xi \wedge *v^\flat = 4g(\xi^\#, v) \text{ vol},$$

so that $\xi^\#, v$ are orthogonal. As G_2 acts transitively on orthonormal pairs, we may assume w.l.o.g. that $\xi = c_1 e_1$, $v = c_2 e_2$, whence

$$\begin{aligned} c_1 e^1 \wedge \alpha^2 &= c\varphi + c_2 \iota_{e_2} * \varphi \\ \Rightarrow 0 &= e^1 \wedge (c\varphi + c_2 \iota_{e_2} * \varphi) \\ &= c(e^{1246} - e^{1257} - e^{1347} - e^{1356}) + c_2(e^{1367} + e^{1345}), \end{aligned}$$

and from this, $c = c_2 = 0$ and hence, $\xi \wedge \alpha^2 = 0$ follows, and this together with $\iota_{\xi^\#} \alpha^2 = 0$ implies that $\alpha^2 = 0$ and hence the injectivity of σ_ξ^2 . \square

We are now ready to calculate the Frölicher-Nijenhuis cohomology algebra of a G_2 -manifold.

Theorem 3.5. *Let (M, φ) be a closed G_2 -manifold, so that $*\varphi \in \Omega^4(M)$ is parallel. Then the differential $\mathcal{L}_{*\varphi}$ is l -regular for all l , and the cohomology algebra $H_{*\varphi}^*(\Omega^*(M))$ is given as follows:*

$$\begin{aligned} H_{*\varphi}^2(M) &= \mathcal{H}^2(M) \oplus \Omega_7^{2;d^*}(M) \oplus \Omega_{14}^{2;d^*}(M) \\ H_{*\varphi}^3(M) &= \mathcal{H}^3(M) \oplus dd^*\Omega_1^3(M) \oplus \Omega_{27}^{3;d}(M) \oplus \Omega_{27}^{3;d^*}(M) \oplus V_{d^*}^3(M) \\ H_{*\varphi}^4(M) &= \mathcal{H}^4(M) \oplus d^*\Omega_1^4(M) \oplus \Omega_{27}^{4;d^*}(M) \oplus \Omega_{27}^{4;d}(M) \oplus V_d^4(M) \\ H_{*\varphi}^5(M) &= \mathcal{H}^5(M) \oplus \Omega_7^{5;d}(M) \oplus \Omega_{14}^{5;d}(M) \end{aligned}$$

with the definitions in (3.6) and (3.12), and $H_{*\varphi}^k(M) = \mathcal{H}^k(M)$ for $k = 0, 1, 6, 7$. Moreover, the summands in this decomposition are L^2 -orthogonal.

Observe that by Corollary 3.4, all summands (apart from $\mathcal{H}^k(M)$) in the above decomposition are infinite dimensional.

Proof. We begin by showing the l -regularity of $\mathcal{L}_{*\varphi}$. For $l < 3$ and $l > 7$, this is obvious as then $\mathcal{L}_{*\varphi,l} = 0$. By Lemma 2.8, $\mathcal{L}_{*\varphi,l}$ is overdetermined elliptic for $l = 3$ and underdetermined elliptic for $l = 7$, whence $\mathcal{L}_{*\varphi}$ is also 3- and 7-regular.

We assert that for $0 \neq \xi \in T_p M$ the symbol $\sigma_\xi(\mathcal{L}_{*\varphi,l}) : \Lambda^{l-3} T_p^* M \rightarrow \Lambda^l T_p^* M$ is injective for $l = 4$ and surjective for $l = 6$. Namely, by (2.37) we have

$$\sigma_\xi(\mathcal{L}_{*\varphi,l})(\alpha^{l-3}) = (\iota_{\xi^\#} * \varphi) \wedge \alpha^{l-3}.$$

Rescaling ξ to a unit vector and using that G_2 acts transitively on the unit sphere, we may assume w.l.o.g. that $\xi = e^1$ for an orthonormal basis (e_i) of $T_p M$ for which (3.3) holds. Now

$$\sigma_{e^1}(\mathcal{L}_{*\varphi,4})(e^1) = (\iota_{e^1} * \varphi) \wedge e^1 = -(e^{1357} - e^{1346} - e^{1256} - e^{1247}) \neq 0$$

$$\sigma_{e^1}(\mathcal{L}_{*\varphi,4})(e^2) = (\iota_{e^1} * \varphi) \wedge e^2 = -(e^{2357} - e^{2346}) \neq 0$$

$$\sigma_{e^1}(\mathcal{L}_{*\varphi,6})(\varphi) = (\iota_{e^1} * \varphi) \wedge \varphi \stackrel{(7.2)}{=} 4 * e^1$$

$$\sigma_{e^1}(\mathcal{L}_{*\varphi,6})(e^{146}) = (\iota_{e^1} * \varphi) \wedge e^{146} = e^{134567} = - * e^2$$

The stabilizer of e_1 in G_2 is isomorphic to $SU(3) \subset G_2$, acting trivially on $\mathbb{R}e_1$ and via the standard 6-dimensional representation on $(e_1)^\perp$. The image and the kernel of $\sigma_{e^1}(\mathcal{L}_{*\varphi,l})$ are invariant under this stabilizer. By the above, $\ker \sigma_{e^1}(\mathcal{L}_{*\varphi,4})$ contains neither e^1 nor e^2 , so by the $SU(3)$ -invariance of the kernel we must have $\ker \sigma_{e^1}(\mathcal{L}_{*\varphi,4}) = 0$. Likewise, the image of $\sigma_{e^1}(\mathcal{L}_{*\varphi,4})$ contains $*e^1$ and $*e^2$, so by the $SU(3)$ -invariance it is all of $*T_p^* M = \Lambda^6 T_p^* M$, showing the above assertion.

Thus, $\mathcal{L}_{*\varphi,l}$ is overdetermined elliptic for $l = 4$ and underdetermined elliptic for $l = 6$, whence $\mathcal{L}_{*\varphi}$ is 4-regular and 6-regular and hence, it is also 5-regular by Theorem 2.7(4). Therefore, the l -regularity of $\mathcal{L}_{*\varphi,l}$ for all l is established, whence by Theorem 2.7(1), $H_{*\varphi}^l(M) = \mathcal{H}_{*\varphi}^l(M)$.

For $l = 0, 7$, $\mathcal{H}_{*\varphi}^l(M) \cong \mathcal{H}^l(M)$ by Proposition 2.10.

For $l = 1$, $H_{*\varphi}^1(M) = \ker \mathcal{L}_{*\varphi}|_{\Omega^1(M)}$. Thus, by Proposition 2.11, $\alpha \in H_{*\varphi}^1(M)$ implies that $\mathcal{L}_{\alpha^\#}(\varphi) = 0$, which in turn implies that $\alpha^\#$ is a Killing vector field. Since a G_2 -manifold is Ricci flat, it follows by Bochner's theorem that $\alpha^\#$ is parallel, whence so is α . In particular, α is harmonic, showing that $H_{*\varphi}^1(M) = \mathcal{H}^1(M)$. For $l = 6$, we have $H_{*\varphi}^6(M) = *H_{*\varphi}^1(M) = *\mathcal{H}^1(M) = \mathcal{H}^6(M)$. This shows that $\mathcal{H}_{*\varphi}^l(M) \cong \mathcal{H}^l(M)$ for $l = 1, 6$.

Next, for $l = 2$, we have $\mathcal{H}_{*\varphi}^2(M)_d = 0$ by (2.32). Thus, we need to determine

$$\mathcal{H}_{*\varphi}^2(M)_{d^*} = \{\alpha^2 \in d^* \Omega^3(M) \mid d^*(\alpha^2 \wedge * \varphi) = 0\}.$$

Observe that $\Omega_7^{2;d^*}(M) \oplus \Omega_{14}^{2;d^*}(M) \subset \mathcal{H}_{*\varphi}^2(M)_{d^*}$. Namely, for $\Omega_{14}^{2;d^*}(M)$ this is obvious as $\alpha^2 \wedge * \varphi = 0$ for $\alpha^2 \in \Omega_{14}^2(M)$. Moreover, by (3.15), we have

for $\Omega_7^{2;d^*}(M) \ni \alpha^2 = \iota_{(df)\#} \varphi$:

$$d^*(\alpha^2 \wedge * \varphi) = d^*(\iota_{(df)\#} \varphi \wedge * \varphi) \stackrel{(7.1)}{=} 3d^* * df = 0.$$

Conversely, let $\alpha^2 \in \mathcal{H}_{*\varphi}^2(M)_{d^*}$. Then there is an $f \in C^\infty(M)$ such that

$$d(\alpha^2 \wedge * \varphi) = \Delta f \text{vol}.$$

Then by the above, $\iota_{(df)\#} \varphi \in \mathcal{H}_{*\varphi}^2(M)_{d^*}$, so that

$$\beta^2 := \alpha^2 + \frac{1}{3} \iota_{(df)\#} \varphi \in \mathcal{H}_{*\varphi}^2(M)_{d^*},$$

and hence,

$$0 = \mathcal{L}_{*\varphi}(\beta^2) = d^*(\beta^2 \wedge * \varphi).$$

On the other hand,

$$\begin{aligned} d(\beta^2 \wedge * \varphi) &= d(\alpha^2 \wedge * \varphi) + \frac{1}{3} d(\iota_{(df)\#} \varphi \wedge * \varphi) \\ &\stackrel{(7.1)}{=} \Delta f \text{vol} + d * df \stackrel{(2.1),(2.3)}{=} \Delta f \text{vol} - \Delta f \text{vol} = 0. \end{aligned}$$

Thus, $\beta^2 \wedge * \varphi \in \mathcal{H}^6(M)$. On the other hand, $\beta^2 \in d^* \Omega^3(M) \subset \Delta \Omega^2(M)$, whence by (2.22), it follows that $\beta^2 \wedge * \varphi \in \Delta \Omega^6(M) = (\mathcal{H}^6(M))^\perp$. Thus, $\beta^2 \wedge * \varphi = 0$, so that $\beta^2 \in \Omega_{14}^{2;d^*}(M)$.

Therefore, $\alpha^2 = -\frac{1}{3} \iota_{(df)\#} \varphi + \beta^2 \in \Omega_7^{2;d^*}(M) \oplus \Omega_{14}^{2;d^*}(M)$, showing that $\mathcal{H}_{*\varphi}^2(M)_{d^*}$ is of the asserted form.

For $l = 3$, it follows from (2.32) that $\mathcal{H}_{*\varphi}^3(M)_d = d\mathcal{H}_{*\varphi}^2(M)_{d^*}$ which by Proposition 3.3 equals $dd^* \Omega_1^3(M) \oplus \Omega_{27}^{3;d}(M)$, whence we need to calculate $\mathcal{H}_{*\varphi}^3(M)_{d^*}$. For this, let $\alpha^3 = d^* \alpha^4$ with $\alpha^4 \in d\Omega^3(M)$. Then

$$(3.23) \quad \alpha^3 \wedge * \varphi = d^* \alpha^4 \wedge * \varphi = (*d * \alpha^4) \wedge * \varphi = d * \alpha^4 \wedge \varphi = d(*\alpha^4 \wedge \varphi),$$

whence

$$\mathcal{L}_{*\varphi}(\alpha^3) = -d^*(\alpha^3 \wedge * \varphi) \stackrel{(3.23)}{=} -d^* d(*\alpha^4 \wedge \varphi),$$

so that $\mathcal{L}_{*\varphi}(\alpha^3) = 0$ iff $0 = d(*\alpha^4 \wedge \varphi) \stackrel{(3.23)}{=} \alpha^3 \wedge * \varphi$. Moreover,

$$\mathcal{L}_{*\varphi}(*\alpha^3) = (d^* * \alpha^3) \wedge * \varphi = (*d\alpha^3) \wedge * \varphi = d\alpha^3 \wedge \varphi = d(\alpha^3 \wedge \varphi).$$

Thus, $\alpha^3 \in \mathcal{H}_{*\varphi}^3(M)_{d^*}$ iff $\mathcal{L}_{*\varphi}(\alpha^3) = \mathcal{L}_{*\varphi}(*\alpha^3) = 0$ iff $\alpha^3 \wedge * \varphi = d(\alpha^3 \wedge \varphi) = 0$, and by (3.17) this is the case iff $\alpha^3 \in \Omega_{27}^{3;d^*}(M) \oplus V_3^{d^*}(M)$.

Again, since $*$: $\mathcal{H}_{*\varphi}^l(M) \rightarrow \mathcal{H}_{*\varphi}^{7-l}(M)$ is an isomorphism, the assertions for $l = 4, 5$ follow as well using Proposition 3.3.

The L^2 -orthogonality of the summands is straightforward by Lemma 3.2 and Proposition 3.3. \square

4. THE FRÖLICHER-NIJENHUIS COHOMOLOGY OF $\text{Spin}(7)$ -MANIFOLDS

In this section, we repeat our discussion for the parallel 4-form on a $\text{Spin}(7)$ -manifold.

Let W be an 8-dimensional oriented real vector space. A $\text{Spin}(7)$ -*structure* on W is a form $\Phi \in \Lambda^4 W^*$ for which there is a positively oriented basis $(e_\mu)_{\mu=0}^7$ of W such that

$$(4.1) \quad \Phi := e^{0123} + e^{0145} + e^{0167} + e^{0246} - e^{0257} - e^{0347} - e^{0356} \\ + e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}.$$

Throughout this section, we shall use Greek indices μ, ν, \dots to run over $0, \dots, 7$, whereas Latin indices i, j, \dots range over $1, \dots, 7$.

If we define the forms φ and $*_7\varphi$ on $V := \text{span}(e_i)_{i=1}^7 \subset W$ as in (3.1) and (3.3), then

$$\Phi = e^0 \wedge \varphi + *_7\varphi.$$

The subgroup of $Gl(W)$ preserving Φ is isomorphic to $\text{Spin}(7)$, which acts irreducibly on W . Since $\text{Spin}(7)$ is compact, there is a unique $\text{Spin}(7)$ -invariant inner product $g = g_\Phi$ on W for which $\|\Phi\|_{g_\Phi}^2 = 14$, and any positively oriented basis (e_μ) of W for which (4.1) holds is orthonormal, whence $\Phi = *\Phi$ when taking the Hodge-* w.r.t. g_Φ .

As before, we denote by W_k the k -dimensional irreducible $\text{Spin}(7)$ -module if there is a unique such module. For instance, W_8 is the irreducible 8-dimensional $\text{Spin}(7)$ -module from above, and $W_8^* \cong W_8$ as there is a $\text{Spin}(7)$ -invariant metric on W_8 . For its exterior powers, we obtain the decompositions

$$(4.2) \quad \begin{aligned} \Lambda^0 W_8 &\cong \Lambda^8 W_8 \cong W_1, & \Lambda^2 W_8 &\cong \Lambda^6 W_8 \cong W_7 \oplus W_{21}, \\ \Lambda^1 W_8 &\cong \Lambda^7 W_8 \cong W_8, & \Lambda^3 W_8 &\cong \Lambda^5 W_8 \cong W_8 \oplus W_{48}, \\ \Lambda^4 W_8 &\cong W_1 \oplus W_7 \oplus W_{27} \oplus W_{35} \end{aligned}$$

where $\Lambda^k W_8 \cong \Lambda^{8-k} W_8$ due to $\text{Spin}(7)$ -invariance of the Hodge-* isomorphism $*$: $\Lambda^k W_8 \rightarrow \Lambda^{8-k} W_8^* \cong \Lambda^{8-k} W_8$. Again, we denote by $\Lambda_l^k W_8 \subset \Lambda^k W_8$ the subspace isomorphic to W_l in the above notation. For instance,

$$\Lambda_8^3 W_8 = \{\iota_v \Phi \mid v \in W_8\} \quad \text{and} \quad \Lambda_{48}^3 W_8 = \{\alpha^3 \in \Lambda^3 W_8 \mid \alpha^3 \wedge \Phi = 0\}.$$

A $\text{Spin}(7)$ -*manifold* is a pair (M, Φ) consisting of an 8-dimensional oriented Riemannian manifold with a 4-form $\Phi \in \Omega^4(M)$ such that at each $p \in M$, Φ_p can be written as in (4.1) w.r.t. some positively oriented basis (e_μ) of $T_p M$ with dual basis (e^μ) of $T_p^* M$, and such that Φ^4 is parallel w.r.t. the Riemannian metric $g = g_\Phi$ on M . Moreover, (4.2) induces a decomposition of differential forms and canonical projections by

$$\Omega_l^k(M) := \Gamma(M, \Lambda_l^k T^* M), \quad \pi_l^k : \Omega^k(M) \longrightarrow \Omega_l^k(M).$$

In analogy to (3.7), we define the spaces

$$(4.3) \quad \begin{aligned} \Omega_l^{k;d}(M) &= \Omega_l^k(M) \cap d\Omega^{k-1}(M) \\ \Omega_l^{k;d^*}(M) &= \Omega_l^k(M) \cap d^*\Omega^{k+1}(M), \end{aligned}$$

and by the same argument as in (3.8) and (3.9), it is immediate that

$$(4.4) \quad \Delta\Omega_l^{k;d}(M) = \Omega_l^{k;d}(M) \quad \text{and} \quad \Delta\Omega_l^{k;d^*}(M) = \Omega_l^{k;d^*}(M)$$

and that the Hodge- $*$ yields isomorphisms $*$: $\Omega_l^{k;d}(M) \xrightarrow{*} \Omega_l^{8-k;d^*}(M)$.

Lemma 4.1. *Let (M, Φ) be a closed $\text{Spin}(7)$ -manifold. Then there is a decomposition*

$$\Omega_{48}^3(M) = \mathcal{H}_{48}^3(M) \oplus \Omega_{48}^{3;d^*}(M) \oplus \pi_{48}^3(d\Omega^3(M)),$$

and $\Omega_{48}^{3;d^*}(M)$ is infinite dimensional.

Proof. In complete analogy the proof of Proposition 3.4, the asserted decomposition will follow if we can show that the differential operator $\phi : \Omega^2(M) \rightarrow \Omega_{48}^3(M) \oplus \Omega^1(M)$, $\alpha^2 \mapsto \pi_{48}^3(d\alpha^2) + d^*\alpha^2$ is overdetermined elliptic. The symbol of ϕ at $0 \neq \xi \in T_p^*M$ is given by

$$\sigma_\xi(\alpha^2) = \pi_{48}^3(\xi \wedge \alpha^2) - \iota_{\xi^\#} \alpha^2,$$

and after rescaling ξ to a unit vector and using the fact that $\text{Spin}(7)$ acts transitively on the unit sphere, we may assume w.l.o.g. that $\xi = e^0$ in an orthonormal basis (e_μ) of T_pM in which (4.1) holds. If $\alpha^2 \in \ker \sigma_{e^0}$, then $\pi_{48}^3(e^0 \wedge \alpha^2) = 0$, whence

$$e^0 \wedge \alpha^2 = \iota_{ce_0+v} \Phi = c\varphi - e^0 \wedge (\iota_v \varphi) + \iota_v * \varphi$$

for some $c \in \mathbb{R}$ and $v \in e_0^\perp$. Thus, $c\varphi + \iota_v * \varphi = 0$, and from this, it easily follows that $c = 0$ and $v = 0$, so that $e^0 \wedge \alpha^2 = 0$. This together with $\iota_{e_0} \alpha^2 = 0$ implies that $\alpha^2 = 0$, so that $\ker \sigma_{e_0} = 0$, showing the claim.

The assertion on the infinite dimension of $\Omega_{48}^{3;d^*}(M)$ follows when applying Lemma 7.4 to the differential operator $\Omega^2(M) \ni \alpha^2 \mapsto \pi_{48}^3(d\alpha^2) \in \Omega_{48}^3(M)$, as $\text{rank}(\Lambda^2 T^*M) = 28 < 48$, and using $\dim \mathcal{H}_{48}^3(M) < \infty$. \square

Theorem 4.2. *Let (M, Φ) be a closed $\text{Spin}(7)$ -manifold. Then the differential \mathcal{L}_Φ is l -regular for all l , and the cohomology algebra $H_\Phi^*(M)$ is given as follows:*

$$\begin{aligned} H_\Phi^3(M) &= \mathcal{H}^3(M) \oplus \Omega_{48}^{3;d^*}(M) \\ H_\Phi^4(M) &= \mathcal{H}^4(M) \oplus d\Omega_{48}^{3;d^*}(M) \oplus d^*\Omega_{48}^{5;d}(M) \\ H_\Phi^5(M) &= \mathcal{H}^5(M) \oplus \Omega_{48}^{5;d}(M) \end{aligned}$$

with the definitions in (4.3), and $H_\Phi^k(M) = \mathcal{H}^k(M)$ for $k = 0, 1, 2, 6, 7, 8$.

Observe that by Lemma 4.1, all summands in this decomposition apart from $\mathcal{H}^k(M)$ are infinite dimensional.

Proof. We begin by showing the l -regularity of $\mathcal{L}_{\Phi;l} : \Omega^{l-3}(M) \rightarrow \Omega^l(M)$ by showing that they are all either underdetermined elliptic or overdetermined elliptic. In order to see this, we first show that the G_2 -equivariant map

$$\wedge\varphi : \Lambda^{3-l}V_7^* \longrightarrow \Lambda^lV_7^*$$

is injective for $l \leq 5$ and surjective for $l \geq 5$.

For $l = 3$, let $0 \neq c \in \mathbb{R} = \Lambda^0V_7^*$. Then $c \wedge \varphi = c\varphi \neq 0$, showing the injectivity. Likewise, for $l = 4$, $\alpha^1 \wedge \varphi \neq 0$ for $0 \neq \alpha^1 \in V_7^*$ is immediate.

For $l = 5$, use the decomposition $\Lambda^2V_7^* = \Lambda_7^2V_7^* \oplus \Lambda_{14}^2V_7^*$, and then Lemma 7.2 immediately implies that the map $\wedge\varphi : \Lambda^2V_7^* \rightarrow \Lambda^5V_7^*$ is an isomorphism, whence both injective and surjective.

For $l = 6$, $(\iota_v * \varphi) \wedge \varphi \stackrel{(7.2)}{=} 4 * v^\flat$ which shows the surjectivity of $\wedge\varphi : \Lambda^3V_7^* \rightarrow \Lambda^6V_7^*$, and for $l = 7$, $\wedge\varphi : \Lambda^4V_7^* \rightarrow \Lambda^7V_7^* = \mathbb{R} \text{ vol}$ is surjective since $*\varphi \wedge \varphi = \|\varphi\|^2 \text{ vol} \neq 0$.

Let $\xi \in T_p^*M$ be a unit vector. Since $\text{Spin}(7)$ acts transitively on the unit sphere, we may choose an orthonormal basis (e_μ) of T_pM for which (4.1) holds such that $\xi = e^0$. Thus, by (2.37) the symbol $\sigma_{e^0}(\mathcal{L}_{\Phi;l}) : \Lambda^{l-3}T_p^*M \rightarrow \Lambda^lT_p^*M$ of $\mathcal{L}_{\Phi;l} : \Omega^{l-3}(M) \rightarrow \Omega^l(M)$ is given w.r.t. this basis as

$$\sigma_{e^0}(\mathcal{L}_{\Phi;l}) : \Lambda^{l-3}W_8^* \longrightarrow \Lambda^lW_8^*, \quad \alpha^{l-3} \longmapsto \varphi \wedge \alpha^{l-3}.$$

As before, let $V_7 := e_0^\perp \subset W_8$. Evidently, the splitting $\Lambda^k W_8^* = \Lambda^k V_7^* \oplus e^0 \wedge \Lambda^{k-1} V_7^*$ is preserved by $\sigma_{e^0}(\mathcal{L}_{\Phi;l})$, and from the above, it now follows that $\sigma_{e^0}(\mathcal{L}_{\Phi;l})$ is injective for $l \leq 5$ and surjective for $l \geq 6$.

Thus, $\mathcal{L}_{\Phi;l}$ is regular for all l , so that by Theorem 2.7 (1), $H_\Phi^l(M) = \mathcal{H}_\Phi^l(M)$.

Since Φ is multi-symplectic, Proposition 2.10 implies that $H_\Phi^l(\Omega^*(M)) = \mathcal{H}_\Phi^l(M)$ for $l = 0, 8$. Moreover, Proposition 2.11 implies that $\alpha^1 \in H_\Phi^1(\Omega^*(M)) = \ker \mathcal{L}_\Phi|_{\Omega^1(M)}$ only if $\mathcal{L}_{(\alpha^1)^\#}(\Phi) = 0$. But every vector field whose flow preserves Φ must be a Killing field, and since $\text{Spin}(7)$ -manifolds are Ricci flat, Bochner's theorem implies that $(\alpha^1)^\#$ is parallel, whence so is α^1 . In particular, α^1 is harmonic, and this shows that $H_\Phi^1(\Omega^*(M)) = \mathcal{H}_\Phi^1(M)$, and $H_\Phi^7(\Omega^*(M)) = *H_\Phi^1(\Omega^*(M)) = *\mathcal{H}_\Phi^1(M) = \mathcal{H}_\Phi^7(M)$.

Next, for $l = 2$, we have $\mathcal{H}_\Phi^2(M)_d = 0$ by (2.32). Thus, we need to determine

$$\mathcal{H}_\Phi^2(M)_{d^*} = \{\alpha^2 \in d^*\Omega^3(M) \mid d^*(\alpha^2 \wedge \Phi) = 0\}.$$

By (4.2), we may decompose $\alpha^2 = \alpha_7^2 + \alpha_{21}^2$, where $\alpha_j \in \Omega_j^2(M)$. By Lemma 7.3 we have

$$(4.5) \quad \alpha_7^2 \wedge \Phi = 3 * \alpha_7^2, \quad \alpha_{21}^2 \wedge \Phi = - * \alpha_{21}^2.$$

Thus,

$$\begin{aligned} d^*(\alpha^2 \wedge \Phi) &= *d * (\alpha^2 \wedge \Phi) = *d(3\alpha_7^2 - \alpha_{21}^2), \\ 3d^*\alpha^2 &= 3 * d * \alpha^2 = *(d(\alpha_7^2 \wedge \Phi - 3\alpha_{21}^2 \wedge \Phi)) = *((d\alpha_7^2 - 3d\alpha_{21}^2) \wedge \Phi). \end{aligned}$$

Thus $\mathcal{L}_\Phi \alpha^2 = 0$ iff $d(3\alpha_7^2 - \alpha_{21}^2) = 0$ and $(d\alpha_7^2 - 3d\alpha_{21}^2) \wedge \Phi = 0$. Substituting $d\alpha_{21}^2 = 3d\alpha_7^2$ from the first into the second equation, it follows that

$$\begin{aligned} 0 &= d\alpha_7^2 \wedge \Phi = d(\alpha_7^2 \wedge \Phi) \stackrel{(4.5)}{=} 3d * \alpha_7^2 \\ 0 &= d\alpha_{21}^2 \wedge \Phi = d(\alpha_{21}^2 \wedge \Phi) \stackrel{(4.5)}{=} -d * \alpha_{21}^2, \end{aligned}$$

so that $d^* \alpha_7^2 = d^* \alpha_{21}^2 = 0$. Thus,

$$d(3\alpha_7^2 - \alpha_{21}^2) = 0 \quad \text{and} \quad d^*(3\alpha_7^2 - \alpha_{21}^2) = 0,$$

whence $3\alpha_7^2 - \alpha_{21}^2$ is harmonic. Since the Laplacian preserves the irreducible decomposition by the Weitzenböck formula (see e.g. [Besse1987, (1.154)]), it follows that α_7^2 and α_{21}^2 are harmonic, whence so is α^2 . Thus, $\mathcal{H}_\Phi^2(M) = \mathcal{H}^2(M)$ and hence, $\mathcal{H}_\Phi^6(M) = \mathcal{H}^6(M)$.

Now let $l = 3$. Since $\mathcal{H}_\Phi^2(M) = \mathcal{H}^2(M)$, it follows from (2.33) that $\mathcal{H}_\Phi^3(M)_d = 0$, whence by (2.31) we only need to determine

$$\mathcal{H}_\Phi^3(M)_{d^*} = \{\alpha^3 \in d^* \Omega^4(M) \mid d^*(\alpha^3 \wedge \Phi) = 0, (d^* * \alpha^3) \wedge \Phi = 0\}.$$

Observe that

$$(d^* * \alpha^3) \wedge \Phi = (*d\alpha^3) \wedge \Phi \stackrel{\Phi=* \Phi}{=} (d\alpha^3) \wedge \Phi = d(\alpha^3 \wedge \Phi)$$

whence $\alpha^3 \in \mathcal{H}_\Phi^3(M)_{d^*}$ iff $\alpha^3 \in d^* \Omega^4(M)$ and $\alpha^3 \wedge \Phi \in \mathcal{H}^7(M)$.

On the other hand, for $\alpha^3 \in d^* \Omega^4(M) \subset \Delta \Omega^4(M)$, (2.22) implies that $\alpha^3 \wedge \Phi \in \Delta(\Omega^7(M)) = (\mathcal{H}^7(M))^\perp$, whence

$$\mathcal{H}_\Phi^3(M)_{d^*} = \{\alpha^3 \in d^* \Omega^4(M) \mid \alpha^3 \wedge \Phi = 0\} = \Omega_{48}^{3;d^*}(M)$$

and hence, $\mathcal{H}_\Phi^3(M)$ is of the asserted form.

Thus, $\mathcal{H}_\Phi^5(M) = *\mathcal{H}_\Phi^3(M) = \mathcal{H}^5(M) \oplus \Omega_{48}^{5;d}(M)$, and by (2.33), $\mathcal{H}_\Phi^4(M)$ is of the asserted form. \square

5. DEFORMATIONS OF G_2 -AND $\text{Spin}(7)$ -STRUCTURES

While Theorems 3.5 and 4.2 give a complete description of $H_{*\varphi}^*(M^7)$ and $H_\Phi^*(M^8)$ for closed G_2 - and $\text{Spin}(7)$ -manifolds, respectively, the cohomology Lie algebras $H_{*\varphi}^*(M^7, TM^7)$ and $H_\Phi^*(M^8, TM^8)$ seem to be more difficult to determine. In this section, we shall use Proposition 2.1 to describe $H_{*\varphi}^k(M, TM)$ and $H_\Phi^k(M, TM)$, respectively, for $k = 0$ and $k = 3$.

Let V be an oriented 7-dimensional vector space, and let $\Lambda_{G_2}^3 V^* \subset \Lambda^3 V^*$ be the set of 3-forms which can be written as in (3.1) for some oriented basis (e^i) of V^* . Likewise, for an oriented 8-dimensional vector space W , we let $\Lambda_{\text{Spin}(7)}^4 W^* \subset \Lambda^4 W^*$ be the set of 4-forms which can be written as in (4.1) for some oriented basis (e^μ) of W^* . By definition, the groups $Gl^+(V)$ and $Gl^+(W)$ of orientation preserving automorphisms of V and W , respectively, act transitively on $\Lambda_{G_2}^3 V^*$ and $\Lambda_{\text{Spin}(7)}^4 W^*$, respectively, so that

$$\Lambda_{G_2}^3 V^* = Gl^+(V)/G_2, \quad \text{and} \quad \Lambda_{\text{Spin}(7)}^4 W^* = Gl^+(W)/\text{Spin}(7).$$

Lemma 5.1. *With V and W as above, the maps*

$$\begin{aligned} \mathfrak{C} : \Lambda_{G_2}^3 V^* &\longrightarrow \Lambda^3 V^* \otimes V, & \varphi &\longmapsto \partial_{g_\varphi}(*_{g_\varphi}\varphi) \\ \mathfrak{P} : \Lambda_{\text{Spin}(7)}^4 W^* &\longrightarrow \Lambda^3 W^* \otimes W, & \Phi &\longmapsto \partial_{g_\Phi}\Phi \end{aligned}$$

are injective immersions, and they are $Gl^+(V)$ - and $Gl^+(W)$ -equivariant, respectively. Here, ∂_g is the map from (2.4), and where as before, g_φ and g_Φ denotes the metric induced by φ and Φ , respectively.

Proof. The equivariance of these maps is immediate from the definition. Thus, as $\Lambda_{G_2}^3 V^*$ and $\Lambda_{\text{Spin}(7)}^4 W^*$ are homogeneous spaces, \mathfrak{C} and \mathfrak{P} are injective immersions iff $\text{Stab}_{Gl^+(7, \mathbb{R})}(\partial_g(*_{g_\varphi}\varphi)) = G_2$ and $\text{Stab}_{Gl^+(8, \mathbb{R})}(\partial_g\Phi) = \text{Spin}(7)$, respectively. The inclusions \supseteq are evident. For the reverse inclusion, observe that the only intermediate Lie groups are given by the following inclusions for any subgroup $\Gamma \subset \mathbb{R}^+ Id$ (see e.g. [Dynkin1952]):

$$(5.1) \quad \begin{array}{ccccc} \Gamma \cdot G_2 & \hookrightarrow & \Gamma \cdot SO(7) & \hookrightarrow & \Gamma \cdot Sl^+(7, \mathbb{R}) \\ \uparrow & & \uparrow & & \uparrow \\ G_2 & \hookrightarrow & SO(7) & \hookrightarrow & Sl(7, \mathbb{R}) \end{array}$$

$$\begin{array}{ccccc} \Gamma \cdot \text{Spin}(7) & \hookrightarrow & \Gamma \cdot SO(8) & \hookrightarrow & \Gamma \cdot Sl^+(8, \mathbb{R}) \\ \uparrow & & \uparrow & & \uparrow \\ \text{Spin}(7) & \hookrightarrow & SO(8) & \hookrightarrow & Sl(8, \mathbb{R}) \end{array}$$

Since $\Lambda^3 V^*$ and V^* ($\Lambda^3 W^*$ and W^* , respectively) are inequivalent irreducible $SO(7)$ -modules ($SO(8)$ -modules, respectively), there is no non-zero element in $\Lambda^3 V^* \otimes V$ (in $\Lambda^3 W^* \otimes W$, respectively) which is invariant under $SO(7)$ ($SO(8)$, respectively). Thus, $\text{Stab}_{Gl^+(V)}\partial_g(*_{g_\varphi}\varphi)$ cannot contain $SO(7)$ and $\text{Stab}_{Gl^+(V)}\partial_g\Phi$ cannot contain $SO(8)$.

Likewise, for $\lambda > 0$, we have $(\lambda Id_V) \cdot \partial_g(*_{g_\varphi}\varphi) = \lambda^{-4}\partial_g(*_{g_\varphi}\varphi)$ and $(\lambda Id_V) \cdot \partial_g\Phi = \lambda^{-4}\partial_g\Phi$, whence $\text{Stab}_{Gl^+(V)}\partial_g(*_{g_\varphi}\varphi) \cap \mathbb{R}^+ Id_V = Id_V$ and $\text{Stab}_{Gl^+(V)}\partial_g\Phi \cap \mathbb{R}^+ Id_W = Id_W$, and from this and (5.1) it follows that $\text{Stab}_{Gl^+(7, \mathbb{R})}(\partial_g(*_{g_\varphi}\varphi)) = G_2$ and $\text{Stab}_{Gl^+(8, \mathbb{R})}(\partial_g\Phi) = \text{Spin}(7)$, which completes the proof. \square

Proposition 5.2. *For a G_2 -manifold (M^7, φ) ,*

$$(5.2) \quad H_{*\varphi}^0(M^7, TM^7) = \{X \in \mathfrak{X}(M^7) \mid \mathcal{L}_X \varphi = 0\}.$$

In particular, if M^7 is closed, then $\dim H_{\varphi}^0(M^7, TM^7) = b^1(M^7)$.*

Likewise, for a $\text{Spin}(7)$ -manifold (M^8, Φ) ,

$$(5.3) \quad H_{*\Phi}^0(M^8, TM^8) = \{X \in \mathfrak{X}(M^8) \mid \mathcal{L}_X \Phi = 0\}.$$

In particular, if M^8 is closed, then $\dim H_{\Phi}^0(M^8, TM^8) = b^1(M^8)$.*

This proposition implies the 4th parts in Theorems 1.1 and 1.2, respectively.

For a closed G_2 - or $\text{Spin}(7)$ -manifold, $b^1(M) = 0$ unless the holonomy of (M, g) is contained in $SU(3) \subsetneq G_2$ or in $G_2 \subsetneq \text{Spin}(7)$, respectively. Thus, for a closed G_2 - or $\text{Spin}(7)$ -manifold with full holonomy, the 0-order cohomology vanishes.

Proof. Let $X \in \mathfrak{X}(M^7)$ be a vector field, $p \in M^7$ and denote by F_X^t the local flow along X , defined in a neighborhood of p . Then because of the pointwise equivariance of \mathfrak{C} we have

$$(F_X^t)^*(\partial_g * \varphi)_{F_X^t(p)} = (F_X^t)^*(\mathfrak{C}(\varphi)_{F_X^t(p)}) = \mathfrak{C}((F_X^t)^*(\varphi_{F_X^t(p)}))$$

and taking the derivative at $t = 0$ yields

$$(5.4) \quad \mathcal{L}_X(\partial_g * \varphi)_p = \mathcal{L}_X(\mathfrak{C}(\varphi))_p = d\mathfrak{C}(\mathcal{L}_X \varphi)_p.$$

Now $\mathcal{L}_X(\partial_g * \varphi) = [X, \partial_g * \varphi]^{FN}$, and since \mathfrak{C} is an immersion by Lemma 5.1, it follows that $X \in H_{*\varphi}^0(M^7, TM^7) = \ker ad_{\partial_g * \varphi}$ iff $\mathcal{L}_X \varphi = 0$, showing (5.2).

Since φ uniquely determines the Riemannian metric g_φ on M^7 , any vector field satisfying $\mathcal{L}_X \varphi = 0$ must be a Killing vector field, and if M^7 is closed, the Ricci flatness of G_2 -manifolds and Bochner's theorem imply that X must be parallel, showing that in this case, $\dim H_\Phi^0(M^7, TM^7) = b^1(M^7)$.

The proof for $\text{Spin}(7)$ -manifolds is completely analogous. \square

Let us now consider deformations of G_2 - and $\text{Spin}(7)$ -structures. Recall that a G_2 -structure on an oriented manifold M^7 is a 3-form $\varphi \in \Omega^3(M^7)$ such that at each $p \in M^7$, φ_p is of the form (3.1) for some oriented basis (e^i) of $T_p^*M^7$. Similarly, a $\text{Spin}(7)$ -structure, on an oriented manifold M^8 is a 4-form $\Phi \in \Omega^4(M^8)$ such that at each $p \in M^8$, Φ_p is of the form (4.1) for some oriented basis (e^μ) of $T_p^*M^8$.

Evidently, a G_2 - and $\text{Spin}(7)$ -structure induces a Riemannian metric on the underlying manifold, but in general, φ and Φ , respectively, need not to be parallel w.r.t. this metric.

Definition 5.3. Let (M^7, φ_0) be a G_2 -manifold. A 3-form $\dot{\varphi}_0 \in \Omega^3(M^7)$ is called a *torsion free infinitesimal deformation* of φ_0 if there exists a family (φ_t) of G_2 -structures which depend fiberwise smoothly on t , such that

$$(5.5) \quad \varphi_t|_{t=0} = \varphi_0, \quad \frac{d}{dt}\bigg|_{t=0} \varphi_t = \dot{\varphi}_0, \quad \text{and} \quad \frac{d}{dt}\bigg|_{t=0} \nabla^{g_t} \varphi_t = 0,$$

where $g_t = g_{\varphi_t}$ is the Riemannian metric induced by φ_t and where the derivatives are taken fiberwise. We call an infinitesimal deformation *trivial* if $\dot{\varphi}_0 = \mathcal{L}_X \varphi_0$ for some vector field $X \in \mathfrak{X}(M^7)$.

For a $\text{Spin}(7)$ -manifold (M^8, Φ_0) , torsion free infinitesimal deformations and trivial deformations of Φ_0 are defined analogously.

For a G_2 -structure φ and the induced Riemannian metric g_φ , there is a section $T \in \Omega^1(M^7, TM^7)$, called the *torsion endomorphism of φ* , (cf. [KLS2017, Proposition 3.3], see also [FG1982]) such that for $v \in TM^7$

$$(5.6) \quad \nabla_v \varphi = \iota_{T(v)} *_{g_\varphi} \varphi.$$

Thus, for $v \in TM$ we have

$$\left. \frac{d}{dt} \right|_{t=0} \nabla_v^{g_t} \varphi_t = \iota_{(\frac{d}{dt}|_0 T^t(v))} (*_{g_0} \varphi_0),$$

whence the last equation in (5.5) is equivalent to

$$(5.7) \quad \left. \frac{d}{dt} \right|_{t=0} T^t = 0,$$

again taking the derivative pointwise.

It was shown in [KLS2017, Proposition 5.1] that there is a bundle isomorphism $\tau : T^*M^7 \otimes TM^7 \rightarrow \Lambda^6 T^*M^7 \otimes TM^7$ such that for $\chi_\varphi := \mathfrak{C}(\varphi) = \partial_{g_\varphi} *_{g_\varphi} \varphi \in \Omega^3(M^7, TM^7)$ we have

$$(5.8) \quad [\chi_\varphi, \chi_\varphi]^{FN} = \tau(T) \in \Omega^6(M, TM),$$

where by abuse of notation we denote by $\tau : \Omega^1(M, TM) \rightarrow \Omega^6(M, TM)$ the pointwise application of τ to sections. Therefore,

$$\begin{aligned} \tau \left(\left. \frac{d}{dt} \right|_{t=0} T^t \right) &= \left. \frac{d}{dt} \right|_{t=0} \tau(T^t) = \left. \frac{d}{dt} \right|_{t=0} [\chi_{\varphi_t}, \chi_{\varphi_t}] = 2 \left[\chi_{\varphi_0}, \left. \frac{d}{dt} \right|_{t=0} \chi_{\varphi_t} \right] \\ &= 2 [\chi_{\varphi_0}, d\mathfrak{C}(\dot{\varphi}_0)], \end{aligned}$$

so that $\dot{\varphi}_0 \in \Omega^3(M^7)$ is a torsion free infinitesimal deformation of φ_0 iff (5.7) holds iff $d\mathfrak{C}(\dot{\varphi}_0) \in \ker(ad_{\chi_{\varphi_0}} : \Omega^3(M^7, TM^7) \rightarrow \Omega^6(M^7, TM^7))$. Since \mathfrak{C} is an immersion and hence $d\mathfrak{C}$ injective by Lemma 5.1, we have an isomorphism

$$\begin{aligned} \{\text{torsion free infinitesimal deformations of } \varphi_0\} &\stackrel{d\mathfrak{C}}{\cong} \\ \ker \left(ad_{\chi_{\varphi_0}} : \Omega^3(M^7, TM^7) \rightarrow \Omega^6(M^7, TM^7) \right) &\cap \Im(d\mathfrak{C}). \end{aligned}$$

Observe that by (5.4)

$$d\mathfrak{C}(\mathcal{L}_X \varphi_0) = \mathcal{L}_X(\mathfrak{C}(\varphi_0)) = [X, \chi_{\varphi_0}]^{FN} = -ad_{\chi_{\varphi_0}}(X),$$

whence there is an induced inclusion

$$\frac{\{\text{torsion free infinitesimal deformations of } \varphi_0\}}{\{\text{trivial deformations of } \varphi_0\}} \xhookrightarrow{d\mathfrak{C}} H_{\varphi_0}^3(M^7, TM^7).$$

The deformations of a torsion free Spin(7)-structure Ψ_0 on an oriented manifold M^8 can be discussed analogously. Here, we get an isomorphism

$$\begin{aligned} \{\text{torsion free infinitesimal deformations of } \Phi_0\} &\stackrel{d\mathfrak{P}}{\cong} \\ \ker \left(ad_{P_{\Phi_0}} : \Omega^3(M^8, TM^8) \rightarrow \Omega^6(M^8, TM^8) \right) &\cap \Im(d\mathfrak{P}), \end{aligned}$$

and for a vector field $X \in \mathfrak{X}(M^8)$ we have

$$d_{\varphi_0} \mathfrak{P}(\mathcal{L}_X \varphi_0) = \mathcal{L}_X(\mathfrak{P}(\Phi_0)) = [X, P_{\Phi_0}]^{FN} = -ad_{P_{\Phi_0}}(X)$$

and hence an injective map

$$\frac{\{\text{torsion free infinitesimal deformations of } \Phi_0\}}{\{\text{trivial deformations of } \Phi_0\}} \hookrightarrow H_{\Phi_0}^3(M^8, TM^8).$$

Thus, what we have shown is the following proposition, which implies the 5th parts in Theorems 1.1 and 1.2, respectively.

Proposition 5.4. *For a G_2 -manifold (M^7, φ) , the cohomology $H_{*\varphi}^3(M^7, TM^7)$ contains the space of all torsion free infinitesimal deformations of φ modulo trivial deformations. In particular, if M^7 is closed then $\dim H_{*\varphi}^3(M^7, TM^7) \geq b^3(M^7) > 0$.*

Likewise, for a $\text{Spin}(7)$ -manifold (M^8, Φ) , the cohomology $H_{\Phi}^3(M^8, TM^8)$ contains the space of all torsion free infinitesimal deformations of Φ modulo trivial deformations. In particular, if M^8 is closed then $\dim H_{\Phi}^3(M^8, TM^8) \geq b_1^4(M^8) + b_7^4(M^8) + b_{35}^4(M^8) > 0$.

The final statements on the relation of deformations and the Betti numbers is due to the regularity theorem of the moduli space of such manifolds due to D. Joyce, see [Joyce1996a], [Joyce2007, Theorem 11.2.8] for the case of G_2 and [Joyce1996b], [Joyce2007, Theorem 11.5.9] for $\text{Spin}(7)$ -manifolds.

6. FUNCTORIAL PROPERTIES OF FRÖLICHER-NIJENHUIS-COHOMOLOGY

In this section we assume that $K \in \Omega^{2k+1}(M, TM)$ is a Maurer-Cartan element of the graded Lie algebra $(\Omega^*(M, TM), [\cdot, \cdot]^{FN})$. Then

$$ad_K : \Omega^*(M, TM) \rightarrow \Omega^*(M, TM)$$

and

$$\mathcal{L}_K : \Omega^*(M) \rightarrow \Omega^*(M)$$

are differentials of the complexes $\Omega^*(M, TM)$ and $\Omega^*(M)$ respectively. The groups

$$H_K^*(M, TM) := H(\Omega^*(M, TM), ad_K) \text{ and } H_K(M) := H(\Omega^*(M), \mathcal{L}_K)$$

will be called *Frölicher-Nijenhuis-cohomology of first kind and second kind respectively*.

To compute the Frölicher-Nijenhuis-cohomology groups we use standard technique of homological algebra [Weibel1994, §5.4].

Proposition 6.1. *Let U and V be two open subsets of M . Then we have the following cohomological long exact sequences (the Meyer-Vietoris sequences)*

$$(6.1) \quad \cdots \rightarrow H_K^i(U \cup V) \xrightarrow{\alpha_*} H_K^i(U) \oplus H_K^i(V) \xrightarrow{\beta_*} H_K^i(U \cap V) \rightarrow H_K^{i+2k+1}(U \cup V) \xrightarrow{\alpha_*} \cdots$$

$$\begin{aligned}
& \cdots \rightarrow H_K^i(U \cup V, T(U \cup V)) \xrightarrow{\alpha_*} H_K^i(U, TU) \oplus H_K^i(V, TV) \xrightarrow{\beta_*} \\
(6.2) \quad & H_K^i(U \cap V, T(U \cap V)) \rightarrow H_K^{i+2k+1}(U \cup V, T(U \cup V)) \xrightarrow{\alpha_*} \cdots
\end{aligned}$$

Proof. We consider the following short exact sequences

$$(6.3) \quad 0 \rightarrow \Omega^*(U \cup V) \xrightarrow{\alpha} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{\beta} \Omega^*(U \cap V) \rightarrow 0,$$

here α is the restriction map, β is the difference map. The long exact sequence (6.1) is a consequence of the short exact sequence (6.3), see e.g. [Weibel1994, Theorem 1.3.1, p. 10]. The second long exact sequence (6.2) is obtained from the following short exact sequence

$$(6.4) \quad 0 \rightarrow \Omega^*(U \cup V, T(U \cup V)) \xrightarrow{\alpha} \Omega^*(U, TU) \oplus \Omega^*(V, TV) \xrightarrow{\beta} \Omega^*(U \cap V, T(U \cap V)) \rightarrow 0.$$

□

Next we shall present some filtrations of the complexes $(\Omega^*(M), ad_K)$ and $(\Omega^*(M, TM), \mathcal{L}_K)$. We shall call $\omega \in \Omega^*(M)$ is *K-stable*, if $\mathcal{L}_K(\omega) = \omega \wedge \alpha$ for some $\alpha \in \Omega^*(M)$. Iterating the action of $\omega \wedge$ on $K^* := \Omega^*(M)$ we have the following filtration

$$(6.5) \quad F^0 K^* = K^* \supset F^1 K^* = \omega \wedge \Omega^*(M) \supset \cdots \supset F^k K^* = \omega^k \wedge \Omega^*(M) \supset \cdots \supset \{0\}.$$

Similarly, an element $\bar{\omega} \in \Omega^*(M, TM)$ will be called *K-stable*, if $[K, \bar{\omega}] = \bar{\omega} \wedge \alpha$ for some $\alpha \in \Omega^*(M)$. Setting $\bar{K}^* := \Omega^*(M, TM)$, we have another filtration

$$(6.6) \quad F^0 \bar{K}^* = \bar{K}^* \supset F^1 \bar{K}^* = \bar{\omega} \wedge \Omega^*(M, TM) \supset \cdots \supset F^k \bar{K}^* = \bar{\omega}^k \wedge \Omega^*(M, TM) \supset \cdots \supset \{0\}.$$

Both the filtrations are bounded, the associated spectral sequences converges to $H_K(M)$ and $H_K(M, TM)$ respectively.

7. APPENDIX

In this appendix, we shall collect some formulas and results which we need in the calculations in this paper.

Lemma 7.1. (cf. [KLS2017, Lemma 6.1]) *For all $u \in V_7$ the following identities hold.*

$$(7.1) \quad * \varphi \wedge (\iota_u \varphi) = 3 * u^\flat$$

$$(7.2) \quad *(u^\flat \wedge \varphi) \wedge \varphi = \varphi \wedge (\iota_u * \varphi) = -4 * u^\flat$$

We also get the following decomposition of $\Lambda^2 V_7$.

Lemma 7.2. (cf. [KLS2017, Lemma 6.2]) *Decompose $\Lambda^2 V_7^* = \Lambda_7^2 V_7^* \oplus \Lambda_{14}^2 V_7^*$ according to (3.5). Then*

$$\begin{aligned}
\Lambda_7^2 V_7^* &= \{\alpha^2 \in \Lambda^2 V_7^* \mid *(\alpha^2 \wedge \varphi) = 2\alpha^2\}, \quad \text{and} \\
\Lambda_{14}^2 V_7^* &= \{\alpha^2 \in \Lambda^2 V_7^* \mid *(\alpha^2 \wedge \varphi) = -\alpha^2\}.
\end{aligned}$$

In particular,

$$(7.3) \quad \begin{aligned} \Lambda_7^2 V_7^* &= \{\alpha^2 + *(\alpha^2 \wedge \varphi) \mid \alpha^2 \in \Lambda^2 V_7^*\}, \quad \text{and} \\ \Lambda_{14}^2 V_7^* &= \{2\alpha^2 - *(\alpha^2 \wedge \varphi) \mid \alpha^2 \in \Lambda^2 V_7^*\}. \end{aligned}$$

The following describes identities of representation of $\text{Spin}(7)$.

Lemma 7.3. (cf. [Kar2005, (4.7), (4.8)]) *Decompose $\Lambda^2 W_8^* = \Lambda_7^2 W_8^* \oplus \Lambda_{21}^2 W_8^*$ according to (4.2). Then*

$$\begin{aligned} \Lambda_7^2 W_8^* &= \{\alpha^2 \in \Lambda^2 W_8^* \mid \Phi \wedge \alpha^2 = 3 * \alpha^2\}, \quad \text{and} \\ \Lambda_{21}^2 W_8^* &= \{\alpha^2 \in \Lambda^2 W_8^* \mid \Phi \wedge \alpha^2 = - * \alpha^2\}. \end{aligned}$$

Finally we also recall the following standard result from the theory of differential operators.

Lemma 7.4. *Let $E \rightarrow M$ and $F \rightarrow M$ be vector bundles, and $\phi : \Gamma(E) \rightarrow \Gamma(F)$ be a linear differential operator.*

If $\text{rank}(E) < \text{rank}(F)$, then the image $\phi(\Gamma(E))$ has infinite codimension in $\Gamma(F)$.

Proof. Let $\phi : \Gamma(E) \rightarrow \Gamma(F)$ be a linear differential operator of degree d , and fix $p \in M$. For $N \in \mathbb{N}$, the N -th Taylor polynomial at p of a section in E and F , respectively, is an element of $P_N(T_p M) \otimes E_p$ and $P_N(T_p M) \otimes F_p$, respectively, where

$$P_N(T_p M) := \bigoplus_{k=0}^N \odot^k(T_p^* M)$$

is the space of polynomials of degree at most N on $T_p M$. The differential operator ϕ then induces for each N a linear map

$$\phi_N : P_{N+d}(T_p M) \otimes E_p \longrightarrow P_N(T_p M) \otimes F_p,$$

associating to the $(N+d)$ -th order Taylor polynomial of $\sigma \in \Gamma(E)$ the N -th order Taylor polynomial of $\phi(\sigma) \in \Gamma(F)$. Since any polynomial in $P_N(T_p M) \otimes F_p$ occurs as the Taylor polynomial of some section of F , it follows that for all $N \in \mathbb{N}$

$$\begin{aligned} \text{codim}(\phi(\Gamma(E)) \subset \Gamma(F)) &\geq \text{codim}(\phi_N(P_{N+d}(T_p M) \otimes E_p) \subset P_N(T_p M) \otimes F_p) \\ &\geq \dim P_N(T_p M) \otimes F_p - \dim P_{N+d}(T_p M) \otimes E_p. \end{aligned}$$

Let $n := \dim M$. Then

$$\lim_{N \rightarrow \infty} \frac{1}{N^n} \dim P_N(T_p M) = \lim_{N \rightarrow \infty} \frac{1}{N^n} \binom{N+n}{n} = \frac{1}{n!},$$

whence

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N^n} (\dim P_N(T_p M) \otimes F_p - \dim P_{N+d}(T_p M) \otimes E_p) \\ = \frac{1}{n!} (\text{rank}(F) - \text{rank}(E)) > 0, \end{aligned}$$

so that $\lim_{N \rightarrow \infty} \dim P_N(T_p M) \otimes F_p - \dim P_{N+d}(T_p M) \otimes E_p = \infty$, and this shows the claim. \square

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